

# CATEGORIFICATION OF HIGHEST WEIGHT MODULES OVER QUANTUM GENERALIZED KAC-MOODY ALGEBRAS

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ABSTRACT. Let  $U_q(\mathfrak{g})$  be a quantum generalized Kac-Moody algebra and let  $V(\Lambda)$  be the integrable highest weight  $U_q(\mathfrak{g})$ -module with highest weight  $\Lambda$ . We prove that the cyclotomic Khovanov-Lauda-Rouquier algebra  $R^\Lambda$  provides a categorification of  $V(\Lambda)$ .

## INTRODUCTION

The *Khovanov-Lauda-Rouquier algebras*, which were introduced independently by Khovanov-Lauda and Rouquier, have emerged as a categorification scheme for quantum groups and their highest weight modules [11, 12, 15]. That is, if  $U_q(\mathfrak{g})$  is the quantum group associated with a symmetrizable Kac-Moody algebra and  $R$  is the corresponding Khovanov-Lauda-Rouquier algebra, then it was shown in [11, 12, 15] that there exists an  $\mathbb{A}$ -algebra isomorphism

$$U_{\mathbb{A}}^-(\mathfrak{g}) \simeq [\mathrm{Proj}(R)] = \bigoplus_{\alpha \in \mathbb{Q}^+} [\mathrm{Proj}(R(\alpha))],$$

where  $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$ ,  $U_{\mathbb{A}}^-(\mathfrak{g})$  is the integral form of  $U_q^-(\mathfrak{g})$ , and  $[\mathrm{Proj}(R)]$  denotes the Grothendieck group of the category  $\mathrm{Proj}(R)$  of finitely generated graded projective  $R$ -modules. Moreover, in [11], Khovanov and Lauda defined a quotient  $R^\Lambda$  of  $R$ , called

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the *cyclotomic Khovanov-Lauda-Rouquier algebra of weight  $\Lambda$* , and conjectured that there exists a  $U_{\mathbb{A}}(\mathfrak{g})$ -module isomorphism

$$V_{\mathbb{A}}(\Lambda) \simeq [\mathrm{Proj}(R^{\Lambda})] = \bigoplus_{\alpha \in \mathbb{Q}^+} [\mathrm{Proj}(R^{\Lambda}(\alpha))],$$

where  $V_{\mathbb{A}}(\Lambda)$  is the integral form of an irreducible highest weight module  $V(\Lambda)$ . It is called the *cyclotomic categorification conjecture*.

Brundan and Stroppel ([2]) proved a special case of this conjecture in type  $A_n$  and, Brundan and Kleshchev ([1]) proved it for type  $A_{\infty}$  and  $A_n^{(1)}$ . In [14], the crystal version of the conjecture was proved. That is, Lauda and Vazirani defined the crystal structure on the set of isomorphism classes of simple objects of the categories  $\mathrm{Rep}(R)$  and  $\mathrm{Rep}(R^{\Lambda})$  of finite-dimensional  $R$ -modules and  $R^{\Lambda}$ -modules, and showed that they are isomorphic to  $B(\infty)$  and  $B(\Lambda)$ , respectively. Recently, Kang and Kashiwara ([7]) proved the cyclotomic categorification conjecture for *all* symmetrizable Kac-Moody algebras. In [16], Webster gave a categorification of tensor products of integrable highest weight modules over quantum groups.

In [10], Kang, Oh and Park introduced a family of *Khovanov-Lauda-Rouquier algebras  $R$  associated with Borchers-Cartan data* and showed that they provide a categorification of quantum generalized Kac-Moody algebras. Moreover, for each dominant integral weight  $\Lambda$ , they defined the *cyclotomic Khovanov-Lauda-Rouquier algebra*

$$R^{\Lambda} = \bigoplus_{\alpha \in \mathbb{Q}^+} R^{\Lambda}(\alpha),$$

where  $R^{\Lambda}(\alpha) = R(\alpha)/I^{\Lambda}(\alpha)$  and  $I^{\Lambda}(\alpha)$  is a two-sided ideal depending on  $\Lambda$ . They proved that the categories of finite-dimensional  $R$ -modules and  $R^{\Lambda}$ -modules have crystal structures that are isomorphic to  $B(\infty)$  and  $B(\Lambda)$ , respectively.

In this paper, we prove that Khovanov-Lauda's cyclotomic categorification conjecture holds for all generalized Kac-Moody algebras. The main result of this paper can be summarized as follows. For each  $i \in I$ , we define two functors

$$\begin{aligned} \mathcal{E}_i^{\Lambda} &: \mathrm{Mod}(R^{\Lambda}(\beta + \alpha_i)) \longrightarrow \mathrm{Mod}(R^{\Lambda}(\beta)), \\ \mathcal{F}_i^{\Lambda} &: \mathrm{Mod}(R^{\Lambda}(\beta)) \longrightarrow \mathrm{Mod}(R^{\Lambda}(\beta + \alpha_i)). \end{aligned}$$

by

$$\begin{aligned}\mathcal{E}_i^\Lambda(N) &= e(\beta, i)N = e(\beta, i)R^\Lambda(\beta + \alpha_i) \otimes_{R^\Lambda(\beta + \alpha_i)} N, \\ \mathcal{F}_i^\Lambda(M) &= q_i^{1 - \langle h_i, \Lambda - \beta \rangle} R^\Lambda(\beta + \alpha_i) e(\beta, i) \otimes_{R^\Lambda(\beta)} M,\end{aligned}$$

where  $q^k$  ( $k \in \mathbb{Z}$ ) denotes the degree shift functor,  $M \in \text{Mod}(R^\Lambda(\beta))$  and  $N \in \text{Mod}(R^\Lambda(\beta + \alpha_i))$ . Then we show that the functors  $\mathcal{E}_i^\Lambda$  and  $\mathcal{F}_i^\Lambda$  are well-defined exact functors on  $\text{Proj}(R^\Lambda)$  (Theorem 4.13) and they satisfy the commutation relations (Theorem 5.2) as operators on  $[\text{Proj}(R^\Lambda)]$

$$[\mathcal{E}_i^\Lambda, \mathcal{F}_j^\Lambda] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where} \quad K_i|_{[\text{Proj}(R^\Lambda(\beta))]} := q_i^{\langle h_i, \Lambda - \beta \rangle}.$$

Therefore, we obtain a categorification of the irreducible highest weight module  $V(\Lambda)$  (Theorem 5.6):

$$[\text{Proj}(R^\Lambda)] \simeq V_{\mathbb{A}}(\Lambda) \quad \text{and} \quad [\text{Rep}(R^\Lambda)] \simeq V_{\mathbb{A}}(\Lambda)^\vee,$$

where  $V_{\mathbb{A}}(\Lambda)^\vee$  is the dual of  $V_{\mathbb{A}}(\Lambda)$  with respect to a non-degenerate symmetric bilinear form on  $V(\Lambda)$ .

We follow the outline given in [7]. The main difference and difficulty in this paper lie in that we need to deal with a family of polynomials  $\mathcal{P}_i$  of degree  $1 - \frac{a_{ii}}{2}$  ( $i \in I$ ) given in (2.1), which makes many of calculations more complicated. Accordingly, the statements in some lemmas and the one in Theorem 4.10 have been modified. The geometric meaning of the polynomials  $\mathcal{P}_i$  was recently clarified when the Borcherds-Cartan datum is symmetric [8].

In [13], Khovanov-Lauda gave a precise description of the relations among the 2-morphisms for categorifications of integrable representations of Kac-Moody algebras, and proved it in the  $\mathfrak{sl}_n$  case. These relations are proved by Cautis-Lauda [4] for symmetrizable Kac-Moody algebras under certain conditions. It would be an interesting problem to adapt their relations to the generalized Kac-Moody algebra case.

This paper is organized as follows. Section 1 contains a brief review of quantum generalized Kac-Moody algebras and their integrable modules. In Section 2, we recall the definition of  $R$  and its basic properties given in [10]. In Section 3, we define the functors  $E_i$ ,  $F_i$  and  $\overline{F}_i$  on  $\text{Mod}(R)$  and derive the relations among them in terms of exact sequences (Theorem 3.5, Theorem 3.9). In Section 4, we show that the structure of  $R^\Lambda$  is compatible with the integrability conditions and the functors  $\mathcal{E}_i^\Lambda$  and  $\mathcal{F}_i^\Lambda$  are well-defined exact functors on  $\text{Proj}(R^\Lambda)$  and  $\text{Rep}(R^\Lambda)$ . In Section 5, by proving the

commutation relations among  $\mathcal{E}_i^\Lambda$  and  $\mathcal{F}_i^\Lambda$ , we conclude that the cyclotomic Khovanov-Lauda-Rouquier algebra  $R^\Lambda$  provides a categorification of the irreducible highest weight module  $V(\Lambda)$  over a quantum generalized Kac-Moody algebra  $U_q(\mathfrak{g})$ .

## 1. QUANTUM GENERALIZED KAC-MOODY ALGEBRAS AND INTEGRABLE MODULES

Let  $I$  be an index set. A square matrix  $\mathbf{A} = (a_{ij})_{i,j \in I}$  with  $a_{ij} \in \mathbb{Z}$  is called a *Borcherds-Cartan matrix* if it satisfies

- (i)  $a_{ii} = 2$  or  $a_{ii} \in 2\mathbb{Z}_{\leq 0}$ , (ii)  $a_{ij} \leq 0$  for  $i \neq j$ , (iii)  $a_{ij} = 0$  if and only if  $a_{ji} = 0$ .

An element  $i$  of  $I$  is said to be *real* if  $a_{ii} = 2$  and *imaginary*, otherwise. We denote by  $I^{\text{re}}$  the set of all real indices and  $I^{\text{im}}$  the set of all imaginary indices. In this paper, we assume that  $\mathbf{A}$  is *symmetrizable*; i.e., there is a diagonal matrix  $\mathbf{D} = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$  such that  $\mathbf{DA}$  is symmetric.

A *Borcherds-Cartan datum*  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$  consists of

- (1) a Borcherds-Cartan matrix  $\mathbf{A}$ ,
- (2) a free abelian group  $\mathbf{P}$ , the *weight lattice*,
- (3)  $\Pi = \{\alpha_i \in \mathbf{P} \mid i \in I\}$ , the set of *simple roots*,
- (4)  $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathbf{P}^\vee := \text{Hom}(\mathbf{P}, \mathbb{Z})$ , the set of *simple coroots*,

satisfying the following properties:

- (a)  $\langle h_i, \alpha_j \rangle = a_{ij}$  for all  $i, j \in I$ ,
- (b)  $\Pi$  is linearly independent,
- (c) for any  $i \in I$ , there exists  $\Lambda_i \in \mathbf{P}$  such that  $\langle h_j, \Lambda_i \rangle = \delta_{ij}$  for all  $j \in I$ .

Let  $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{P}^\vee$ . Since  $\mathbf{A}$  is symmetrizable, there is a symmetric bilinear form  $(\mid)$  on  $\mathfrak{h}^*$  satisfying

$$(\alpha_i \mid \alpha_j) = d_i a_{ij} \quad \text{and} \quad (\alpha_i \mid \lambda) = d_i \langle h_i, \lambda \rangle \quad \text{for all } i, j \in I, \lambda \in \mathfrak{h}^*.$$

We denote by  $\mathbf{P}^+ := \{\lambda \in \mathbf{P} \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, i \in I\}$  the set of *dominant integral weights*. The free abelian group  $\mathbf{Q} = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$  is called the *root lattice*. Set  $\mathbf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ . For  $\alpha = \sum k_i \alpha_i \in \mathbf{Q}^+$  and  $i \in I$ , we define

$$\text{Supp}(\alpha) = \{i \in I \mid k_i \neq 0\}, \quad \text{Supp}_i(\alpha) = k_i, \quad |\alpha| = \sum_{i \in I} k_i.$$

Let  $q$  be an indeterminate and  $m, n \in \mathbb{Z}_{\geq 0}$ . Set  $q_i = q^{d_i}$  for  $i \in I$ . If  $i \in I^{\text{re}}$ , define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \left[ \begin{matrix} m \\ n \end{matrix} \right]_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

**Definition 1.1.** The *quantum generalized Kac-Moody algebra*  $U_q(\mathfrak{g})$  associated with a Borcherds-Cartan datum  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$  is the associative algebra over  $\mathbb{Q}(q)$  with **1** generated by  $e_i, f_i$  ( $i \in I$ ) and  $q^h$  ( $h \in \mathbf{P}^\vee$ ) satisfying following relations:

- (i)  $q^0 = 1$ ,  $q^h q^{h'} = q^{h+h'}$  for  $h, h' \in \mathbf{P}^\vee$ ,
- (ii)  $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$ ,  $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$  for  $h \in \mathbf{P}^\vee, i \in I$ ,
- (iii)  $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ , where  $K_i = q_i^{h_i}$ ,
- (iv)  $\sum_{r=0}^{1-a_{ij}} \left[ \begin{matrix} 1-a_{ij} \\ r \end{matrix} \right]_i e_i^{1-a_{ij}-r} e_j e_i^r = 0$  if  $i \in I^{\text{re}}$  and  $i \neq j$ ,
- (v)  $\sum_{r=0}^{1-a_{ij}} \left[ \begin{matrix} 1-a_{ij} \\ r \end{matrix} \right]_i f_i^{1-a_{ij}-r} f_j f_i^r = 0$  if  $i \in I^{\text{re}}$  and  $i \neq j$ ,
- (vi)  $e_i e_j - e_j e_i = 0$ ,  $f_i f_j - f_j f_i = 0$  if  $a_{ij} = 0$ .

Let  $U_q^+(\mathfrak{g})$  (resp.  $U_q^-(\mathfrak{g})$ ) be the subalgebra of  $U_q(\mathfrak{g})$  generated by the elements  $e_i$  (resp.  $f_i$ ).

**Definition 1.2.** We define  $\mathcal{O}_{\text{int}}$  to be the category consisting of  $U_q(\mathfrak{g})$ -modules  $V$  satisfying the following properties:

- (i)  $V$  has a *weight decomposition* with finite-dimensional weight spaces; i.e.,

$$V = \bigoplus_{\mu \in \mathbf{P}} V_\mu \quad \text{with} \quad \dim V_\mu < \infty,$$

where  $V_\mu = \{v \in V \mid q^h v = q^{\langle h, \mu \rangle} v \text{ for all } h \in \mathbf{P}^\vee\}$ ,

- (ii) there are finitely many  $\lambda_1, \dots, \lambda_s \in \mathbf{P}$  such that

$$\text{wt}(V) := \{\mu \in \mathbf{P} \mid V_\mu \neq 0\} \subset \bigcup_{i=1}^s (\lambda_i - \mathbf{Q}^+),$$

- (iii) the action of  $f_i$  on  $V$  is locally nilpotent for  $i \in I^{\text{re}}$ ,
- (iv) if  $i \in I^{\text{im}}$ , then  $\langle h_i, \mu \rangle \in \mathbb{Z}_{\geq 0}$  for all  $\mu \in \text{wt}(V)$ ,
- (v) if  $i \in I^{\text{im}}$  and  $\langle h_i, \mu \rangle = 0$ , then  $f_i V_\mu = 0$ ,
- (vi) if  $i \in I^{\text{im}}$  and  $\langle h_i, \mu \rangle \leq -a_{ii}$ , then  $e_i V_\mu = 0$ .

For  $\Lambda \in \mathbf{P}$ , a  $U_q(\mathfrak{g})$ -module  $V$  is called a *highest weight module* with *highest weight*  $\Lambda$  and *highest weight vector*  $v_\Lambda$  if there exists  $v_\Lambda \in V$  such that

$$(1) V = U_q(\mathfrak{g})v_\Lambda, \quad (2) q^h v_\Lambda = q^{\langle h, \Lambda \rangle} v_\Lambda \text{ for all } h \in \mathbf{P}^\vee, \quad (3) e_i v_\Lambda = 0 \text{ for all } i \in I.$$

For  $\Lambda \in \mathbf{P}^+$ , let us denote by  $V(\Lambda)$  the  $U_q(\mathfrak{g})$ -module generated by  $v_\Lambda$  with the defining relation:

- (a)  $v_\Lambda$  is a highest weight vector of weight  $\Lambda$ ,
- (b)  $f_i^{\langle h_i, \Lambda \rangle + 1} v_\Lambda = 0$  for any  $i \in I^{\text{re}}$ ,
- (c)  $f_i v_\Lambda = 0$  if  $\langle h_i, \Lambda \rangle = 0$ .

**Proposition 1.3** ([3, 5, 6]).

- (i) For any  $\Lambda \in \mathbf{P}^+$ ,  $V(\Lambda)$  is an irreducible  $U_q(\mathfrak{g})$ -module.
- (ii) If  $V$  is a highest weight module in  $\mathcal{O}_{\text{int}}$ , then  $V$  is isomorphic to  $V(\Lambda)$  for some  $\Lambda \in \mathbf{P}^+$ .
- (iii) Any module in  $\mathcal{O}_{\text{int}}$  is semisimple.

Let  $\phi$  be the anti-automorphism of  $U_q(\mathfrak{g})$  given by

$$\phi(e_i) = f_i, \quad \phi(f_i) = e_i \quad \text{and} \quad \phi(q^h) = q^h.$$

In [9], it was shown that there exists a unique non-degenerate symmetric bilinear form  $(\ , \ )$  on  $V(\Lambda)$  ( $\Lambda \in \mathbf{P}^+$ ) satisfying

$$(1.1) \quad (v_\Lambda, v_\Lambda) = 1, \quad (xu, v) = (u, \phi(x)v) \text{ for } x \in U_q(\mathfrak{g}) \text{ and } u, v \in V(\Lambda).$$

Set  $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$ . We define the  $\mathbb{A}$ -form  $V_{\mathbb{A}}(\Lambda)$  of  $V(\Lambda)$  to be

$$V_{\mathbb{A}}(\Lambda) = U_{\mathbb{A}}(\mathfrak{g})v_\Lambda,$$

where  $U_{\mathbb{A}}(\mathfrak{g})$  is the  $\mathbb{A}$ -subalgebra of  $U_q(\mathfrak{g})$  defined in [5, Section 9].

The dual of  $V_{\mathbb{A}}(\Lambda)$  is defined to be

$$V_{\mathbb{A}}(\Lambda)^\vee = \{v \in V(\Lambda) \mid (u, v) \in \mathbb{A} \text{ for all } u \in V_{\mathbb{A}}(\Lambda)\}.$$

## 2. THE KHOVANOV-LAUDA-ROUQUIER ALGEBRAS FOR GENERALIZED KAC-MOODY ALGEBRAS

We take a graded commutative ring  $\mathbf{k} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbf{k}_n$  as a base ring. For a given Borchers-Cartan datum  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$ , we take  $\mathcal{Q}_{i,j}(u, v) (i, j \in I)$  in  $\mathbf{k}[u, v]$  such that  $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$  and  $\mathcal{Q}_{i,j}(u, v)$  has the form

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ \sum_{d_i p + d_j q \leq -(\alpha_i | \alpha_j)} t_{i,j}^{p,q} u^p v^q & \text{if } i \neq j, \end{cases}$$

where  $t_{i,j}^{-a_{ij},0} \in \mathbf{k}_0^\times$  and  $t_{i,j}^{p,q} \in \mathbf{k}_{-2((\alpha_i | \alpha_j) + d_i p + d_j q)}$  with  $t_{i,j}^{p,q} = t_{j,i}^{q,p}$ .

For all  $i \in I$ , we take polynomials  $\mathcal{P}_i(u, v)$  in  $\mathbf{k}[u, v]$  which have the form

$$(2.1) \quad \mathcal{P}_i(u, v) = \sum_{p+q \leq 1 - \frac{a_{ii}}{2}} w_i^{p,q} u^p v^q,$$

where  $w_i^{1 - \frac{a_{ii}}{2}, 0}, w_i^{0, 1 - \frac{a_{ii}}{2}} \in \mathbf{k}_0^\times$  and  $w_i^{p,q} \in \mathbf{k}_{2d_i(1-p-q-\frac{a_{ii}}{2})}$ .

**Remark 2.1.** In [10], it was assumed that  $\mathcal{P}_i(u, v)$  is a symmetric homogeneous polynomial. But, in this paper, we do not assume that  $\mathcal{P}_i(u, v)$  is symmetric. Instead, we put more restrictions on the leading terms of  $\mathcal{P}_i(u, v)$ . Accordingly, the defining relations of Khovanov-Lauda-Rouquier algebras in Definition 2.2 below are modified from the ones in [10]. This choice will be used in a critical way in the proof of Lemma 4.3 and Lemma 5.5. The main results of [10] are still valid after this modification.

We denote by  $S_n = \langle s_1, \dots, s_{n-1} \rangle$  the symmetric group on  $n$  letters, where  $s_i = (i, i+1)$  is the transposition. Then  $S_n$  acts on  $I^n$  and  $\mathbf{k}[x_1, \dots, x_n]$  in a natural way.

We define the operator  $\partial_a$  on  $\mathbf{k}[x_1, \dots, x_n]$ , by

$$\partial_{a,b} f = \frac{s_{a,b} f - f}{x_a - x_b}, \quad \partial_a := \partial_{a,a+1},$$

where  $s_{a,b} = (a, b)$  is the transposition.

For the sake of simplicity, we assume that  $I$  is a finite set.

**Definition 2.2** ([10]). The Khovanov-Lauda-Rouquier algebra  $R(n)$  of degree  $n$  associated with the data  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$ ,  $(\mathcal{Q}_{i,j})_{i,j \in I}$  and  $(\mathcal{P}_i)_{i \in I}$  is the associative algebra over  $\mathbf{k}$  generated by  $e(\nu)$  ( $\nu \in I^n$ ),  $x_k$  ( $1 \leq k \leq n$ ),  $\tau_\ell$  ( $1 \leq \ell \leq n-1$ ) with following

relations:

$$\begin{aligned}
(2.2) \quad & e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^n} e(\nu) = 1, \\
& x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu) x_k, \\
& \tau_\ell e(\nu) = e(s_\ell \nu) \tau_\ell, \quad \tau_k \tau_\ell = \tau_\ell \tau_k \text{ if } |k - \ell| > 1, \\
& \tau_k^2 e(\nu) = \begin{cases} (\partial_k \mathcal{P}_{\nu_k}(x_k, x_{k+1})) \tau_k e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \mathcal{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), & \text{if } \nu_k \neq \nu_{k+1}. \end{cases} \\
(2.3) \quad & (\tau_k x_\ell - x_{s_k(\ell)} \tau_k) e(\nu) = \begin{cases} -\mathcal{P}_{\nu_k}(x_k, x_{k+1}) e(\nu) & \text{if } \ell = k, \nu_k = \nu_{k+1}, \\ \mathcal{P}_{\nu_k}(x_k, x_{k+1}) e(\nu) & \text{if } \ell = k+1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
(2.4) \quad & (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) \\
& = \begin{cases} \mathcal{P}_{\nu_k}(x_k, x_{k+2}) \overline{\mathcal{Q}}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2}) e(\nu) & \text{if } \nu_k = \nu_{k+2} \neq \nu_{k+1}, \\ \overline{\mathcal{P}}'_{\nu_k}(x_k, x_{k+1}, x_{k+2}) \tau_k e(\nu) + \overline{\mathcal{P}}''_{\nu_k}(x_k, x_{k+1}, x_{k+2}) \tau_{k+1} e(\nu) & \text{if } \nu_k = \nu_{k+1} = \nu_{k+2}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\overline{\mathcal{P}}'_i(u, v, w) &= \overline{\mathcal{P}}'_i(v, u, w) := \frac{\mathcal{P}_i(v, u) \mathcal{P}_i(u, w)}{(u-v)(u-w)} + \frac{\mathcal{P}_i(u, w) \mathcal{P}_i(v, w)}{(u-w)(v-w)} - \frac{\mathcal{P}_i(u, v) \mathcal{P}_i(v, w)}{(u-v)(v-w)}, \\
\overline{\mathcal{P}}''_i(u, v, w) &= \overline{\mathcal{P}}''_i(u, w, v) := -\frac{\mathcal{P}_i(u, v) \mathcal{P}_i(u, w)}{(u-v)(u-w)} - \frac{\mathcal{P}_i(u, w) \mathcal{P}_i(w, v)}{(u-w)(v-w)} + \frac{\mathcal{P}_i(u, v) \mathcal{P}_i(v, w)}{(u-v)(v-w)}, \\
\overline{\mathcal{Q}}_{i,j}(u, v, w) &:= \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u-w}.
\end{aligned}$$

The  $\mathbb{Z}$ -grading on  $R(n)$  is given by

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = 2d_{\nu_k}, \quad \deg(\tau_\ell e(\nu)) = -(\alpha_{\nu_\ell} | \alpha_{\nu_{\ell+1}})$$

for all  $\nu \in I^n$ ,  $1 \leq k \leq n$  and  $1 \leq \ell < n$ .

For  $\nu = (\nu_1, \dots, \nu_n) \in I^n$  and  $1 \leq m \leq n$ , we define

$$\begin{aligned}
\nu_{<m} &= (\nu_1, \dots, \nu_{m-1}), & \nu_{\leq m} &= (\nu_1, \dots, \nu_m), \\
\nu_{>m} &= (\nu_{m+1}, \dots, \nu_n), & \nu_{\geq m} &= (\nu_m, \dots, \nu_n).
\end{aligned}$$



For pairwise distinct  $a, b, c \in \{1, \dots, n\}$ , let us define

$$\begin{aligned} e_{a,b} &= \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_b}} e(\nu), \quad \mathcal{P}_{a,b} = \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_b}} \mathcal{P}_{\nu_a}(x_a, x_b) e(\nu), \\ \overline{\mathcal{Q}}_{a,b,c} &= \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_c \neq \nu_b}} \frac{\mathcal{Q}_{\nu_a, \nu_b}(x_a, x_b) - \mathcal{Q}_{\nu_a, \nu_b}(x_c, x_b)}{x_a - x_c} e(\nu), \quad \overline{\mathcal{Q}}_a := \overline{\mathcal{Q}}_{a, a+1, a+2}, \\ \overline{\mathcal{P}}'_{a,b,c} &= \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_b = \nu_c}} \overline{\mathcal{P}}'_{\nu_a}(x_a, x_b, x_c) e(\nu), \quad \overline{\mathcal{P}}'_a := \overline{\mathcal{P}}'_{a, a+1, a+2}, \\ \overline{\mathcal{P}}''_{a,b,c} &= \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_b = \nu_c}} \overline{\mathcal{P}}''_{\nu_a}(x_a, x_b, x_c) e(\nu), \quad \overline{\mathcal{P}}''_a := \overline{\mathcal{P}}''_{a, a+1, a+2}. \end{aligned}$$

Then we have

$$\tau_{a+1} \tau_a \tau_{a+1} - \tau_a \tau_{a+1} \tau_a = \overline{\mathcal{Q}}_a \mathcal{P}_{a, a+2} + \overline{\mathcal{P}}'_a \tau_a + \overline{\mathcal{P}}''_a \tau_{a+1}.$$

Note that we have  $\overline{\mathcal{P}}'_a \tau_a = \tau_a \overline{\mathcal{P}}'_a$  and  $\overline{\mathcal{P}}''_a \tau_{a+1} = \tau_{a+1} \overline{\mathcal{P}}''_a$  by the formula (2.5) below.

We define the operator, also denoted by  $\partial_{a,b}$ , on  $\oplus_{\nu \in I^n} \mathbf{k}[x_1, \dots, x_n] e(\nu)$ , by

$$\partial_{a,b} f = \frac{s_{a,b} f - f}{x_a - x_b} e_{a,b}, \quad \partial_a := \partial_{a, a+1}.$$

Then we obtain

$$(2.5) \quad \tau_a f - (s_a f) \tau_a = f \tau_a - \tau_a (s_a f) = (\partial_a f) \mathcal{P}_{a, a+1}.$$

For  $\beta \in \mathbf{Q}^+$  with  $|\beta| = n$ , we set

$$I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \dots + \alpha_{\nu_n} = \beta\}.$$

We define

$$\begin{aligned} R(m, n) &= R(m) \otimes_{\mathbf{k}} R(n) \subset R(m+n), \\ e(n) &= \sum_{\nu \in I^n} e(\nu), \quad e(\beta) = \sum_{\nu \in I^\beta} e(\nu), \quad R(\beta) = e(\beta) R(n), \\ e(n, i) &= \sum_{\substack{\nu \in I^{n+1}, \\ \nu_{n+1} = i}} e(\nu), \quad e(i, n) = \sum_{\substack{\nu \in I^{n+1}, \\ \nu_1 = i}} e(\nu), \\ e(\beta, i) &= e(\beta + \alpha_i) e(n, i), \quad e(i, \beta) = e(\beta + \alpha_i) e(i, n). \end{aligned}$$

Then  $R = \bigoplus_{\alpha \in \mathbf{Q}^+} R(\alpha)$ .

**Proposition 2.3** ([10]).

- (i)  $R(\alpha)$  is noetherian.
- (ii) There are only finitely many irreducible graded  $R(\alpha)$ -modules up to isomorphism and grading shift. Moreover, all the irreducible graded  $R(\alpha)$ -modules are finite-dimensional.
- (iii) The Krull-Schmidt unique direct sum decomposition property holds for all finitely generated graded  $R(\alpha)$ -modules.

In the rest of this section, assume that  $\mathbf{k}_0$  is a field. Let  $\text{Mod}(R(\alpha))$  (resp.  $\text{Proj}(R(\alpha))$ ,  $\text{Rep}(R(\alpha))$ ) be the category of arbitrary (resp. finitely generated projective, finite-dimensional over  $\mathbf{k}_0$ ) graded left  $R(\alpha)$ -modules. The morphisms in these categories are degree preserving homomorphisms. Define

$$[\text{Proj}(R)] := \bigoplus_{\alpha \in \mathbb{Q}^+} [\text{Proj}(R(\alpha))] \text{ and } [\text{Rep}(R)] := \bigoplus_{\alpha \in \mathbb{Q}^+} [\text{Rep}(R(\alpha))],$$

where  $[\text{Proj}(R(\alpha))]$  (resp.  $[\text{Rep}(R(\alpha))]$ ) is the Grothendieck group of  $\text{Proj}(R(\alpha))$  (resp.  $\text{Rep}(R(\alpha))$ ). We can define the degree shift functors  $q^m$  ( $m \in \mathbb{Z}$ ) on  $\text{Mod}(R(\alpha))$  given as follows: For  $M = \bigoplus_{k \in \mathbb{Z}} M_k$ ,

$$q^m(M) := M\langle -m \rangle \quad \text{where } M\langle m \rangle_k = M_{k+m}.$$

Then one can define  $\mathbb{A}$ -module structures on  $[\text{Proj}(R)]$  and  $[\text{Rep}(R)]$ . The following theorem provides a categorification of quantum generalized Kac-Moody algebras.

**Theorem 2.4** ([10]). *There is an injective  $\mathbb{A}$ -algebra homomorphism*

$$U_{\mathbb{A}}^-(\mathfrak{g}) \hookrightarrow [\text{Proj}(R)].$$

*It is an isomorphism if  $a_{ii} \neq 0$  for any  $i \in I$ .*

### 3. THE FUNCTORS $E_i$ AND $F_i$ ON $\text{Mod}(R)$ .

From the natural embedding  $R(\beta) \otimes_{\mathbf{k}} R(\alpha_i) \hookrightarrow R(\beta + \alpha_i)$ , we obtain the functors

$$E_i: \text{Mod}(R(\beta + \alpha_i)) \rightarrow \text{Mod}(R(\beta)),$$

$$F_i: \text{Mod}(R(\beta)) \rightarrow \text{Mod}(R(\beta + \alpha_i))$$

given by

$$\begin{aligned} E_i(M) &= M \mapsto e(\beta, i)M \simeq e(\beta, i)R(\beta + \alpha_i) \otimes_{R(\beta + \alpha_i)} M, \\ F_i(N) &= N \mapsto R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} N \end{aligned}$$

for  $M \in \text{Mod}(R(\beta + \alpha_i))$  and  $N \in \text{Mod}(R(\beta))$ .

Let  $\xi_n: R(n) \rightarrow R(n+1)$  be the algebra monomorphism given by

$$\xi_n(x_k) = x_{k+1}, \quad \xi_n(\tau_\ell) = \tau_{\ell+1}, \quad \xi_n(e(\nu)) = \sum_{i \in I} e(i, \nu)$$

for all  $1 \leq k \leq n$ ,  $1 \leq \ell < n$  and  $\nu \in I^n$ . Let  $R^1(n)$  be the image of  $\xi_n$ . Then for each  $i \in I$ , we can define the functor

$$\overline{F}_i: \text{Mod}(R(\beta)) \rightarrow \text{Mod}(R(\beta) + \alpha_i) \text{ by } N \mapsto R(\beta + \alpha_i)e(i, \beta) \otimes_{R(\beta)} N.$$

Here, the right  $R(\beta)$ -module structure on  $R(\beta + \alpha_i)e(i, \beta)$  is given by the embedding

$$R(\beta) \xrightarrow{\sim} R^1(\beta) \hookrightarrow R(\beta + \alpha_i).$$

From now on, we will investigate the relationship among these functors.

**Proposition 3.1** ([10, Corollary 2.5]). *We have a decomposition*

$$R(n+1) = \bigoplus_{a=1}^{n+1} R(n, 1)\tau_n \cdots \tau_a = \bigoplus_{a=1}^{n+1} R(n) \otimes \mathbf{k}[x_{n+1}]\tau_n \cdots \tau_a.$$

Furthermore,  $R(n+1)$  is a free  $R(n, 1)$ -module of rank  $n+1$ .

**Lemma 3.2.** *For  $1 \leq a \leq n$ ,  $f(x_n) \in \mathbf{k}[x_n]$  and  $y \in R(n)$ , we have*

$$\tau_a \cdots \tau_{n-1} f(x_n) \tau_n y \equiv \tau_a \cdots \tau_{n-1} \tau_n f(x_{n+1}) y \pmod{R(n, 1)}.$$

*Proof.* By (2.5), we have

$$(3.1) \quad \tau_a \cdots \tau_{n-1} f(x_n) \tau_n y = \tau_a \cdots \tau_{n-1} (\tau_n f(x_{n+1}) + (\partial_n f) \mathcal{P}_{n, n+1}) y.$$

Since  $(\partial_n f) \mathcal{P}_{n, n+1} \in \sum_{\nu \in I^{n+1}} \mathbf{k}[x_n, x_{n+1}]e(\nu) \subset R(n, 1)$ , the second term in the right-hand side of (3.1) is equal to

$$\tau_a \cdots \tau_{n-1} (\partial_n f) \mathcal{P}_{n, n+1} y \equiv 0 \pmod{R(n, 1)}.$$

Hence our assertion holds.  $\square$

**Proposition 3.3.** *The homomorphism  $R(n) \otimes_{R(n-1)} R(n) \longrightarrow R(n+1)$  given by*

$$x \otimes y \longmapsto x\tau_n y \quad (x, y \in R(n))$$

*induces an isomorphism of  $(R(n), R(n))$ -bimodules*

$$(3.2) \quad R(n) \otimes_{R(n-1)} R(n) \oplus R(n, 1) \xrightarrow{\sim} R(n+1).$$

*Proof.* Using Lemma 3.2, we can apply a similar argument given in [7, Proposition 3.3]  $\square$

**Corollary 3.4.** *There exists a natural isomorphism*

$$\begin{aligned} & e(n, i)R(n+1)e(n, j) \\ &= \begin{cases} q_i^{-a_{ij}} R(n)e(n-1, j) \otimes_{R(n-1)} e(n-1, i)R(n) & \text{if } i \neq j, \\ q_i^{-a_{ii}} R(n)e(n-1, i) \otimes_{R(n-1)} e(n-1, i)R(n) \oplus e(n, i)R(n, i)e(n, i) & \text{if } i = j, \end{cases} \end{aligned}$$

where  $q$  is the degree shift functor and  $q_i = q^{d_i}$ .

*Proof.* By applying the exact functor  $e(n, i) \bullet e(n, j)$  on (3.2), we obtain

$$\begin{aligned} & e(n, i)R(n+1)e(n, j) \cong e(n, i)(R(n) \otimes_{R(n-1)} R(n) \oplus R(n, 1))e(n, j) \\ & \cong e(n, i)R(n) \otimes_{R(n-1)} R(n)e(n, j) \oplus \delta_{ij}e(n, i)R(n, 1)e(n, j) \\ & \cong R(n)e(n, i) \otimes_{R(n-1)} e(n, j)R(n) \oplus \delta_{ij}e(n, i)R(n, 1)e(n, j) \\ & \cong R(n)e(n-1, j) \otimes_{R(n-1)} e(n-1, i)R(n) \oplus \delta_{ij}e(n, i)R(n, 1)e(n, j). \end{aligned}$$

The grading-shift  $q_i^{-a_{ij}} = q^{-(\alpha_i|\alpha_j)}$  arises from  $e(n, i)\tau_n e(n, j)$ .  $\square$

Note that the kernels of  $E_i F_j$  and  $F_j E_i$  are given by

$$(3.3) \quad \begin{aligned} & e(n, i)R(n+1)e(n, j)e(\beta) = e(\beta, i)R(\beta + \alpha_j)e(\beta, j), \\ & R(n)e(n-1, j) \otimes_{R(n-1)} e(n-1, i)R(n)e(\beta) = R(\beta - \alpha_i + \alpha_j)e(\beta - \alpha_i, j), \end{aligned}$$

respectively. The following theorem is an immediate consequence of Corollary 3.4.

**Theorem 3.5.** *There exist natural isomorphisms*

$$E_i F_j \xrightarrow{\sim} \begin{cases} q_i^{-a_{ij}} F_j E_i & \text{if } i \neq j, \\ q_i^{-a_{ii}} F_i E_i \oplus \text{Id} \otimes \mathbf{k}[t_i] & \text{if } i = j, \end{cases}$$

where  $t_i$  is an indeterminate of degree  $2d_i$  and  $\text{Id} \otimes \mathbf{k}[t_i]: \text{Mod}(R(\beta)) \rightarrow \text{Mod}(R(\beta))$  is the functor  $M \mapsto M \otimes \mathbf{k}[t_i]$ .

**Proposition 3.6.** *There exists an injective homomorphism*

$$\Phi: R(n) \otimes_{R^1(n-1)} R^1(n) \rightarrow R(n+1) \quad \text{given by } x \otimes y \mapsto xy.$$

Moreover, its image  $R(n)R^1(n)$  has decomposition

$$R(n)R^1(n) = \bigoplus_{a=2}^{n+1} R(n, 1)\tau_n \cdots \tau_2 = \bigoplus_{a=0}^{n-1} \tau_a \cdots \tau_1 R(1, n).$$

*Proof.* The proof is the same as that of [7, Proposition 3.7] □

By Proposition 3.6, there exists a map  $\varphi_1: R(n+1) \rightarrow R(n) \otimes \mathbf{k}[x_{n+1}]$  given by

$$(3.4) \quad \begin{aligned} R(n+1) \rightarrow \text{Coker}(\Phi) &\cong \frac{\bigoplus_{a=1}^{n+1} R(n, 1)\tau_n \cdots \tau_a}{\bigoplus_{a=2}^{n+1} R(n, 1)\tau_n \cdots \tau_a} \xleftarrow{\sim} R(n, 1)\tau_n \cdots \tau_1 \xleftarrow{\sim} R(n, 1) \\ &\cong R(n) \otimes \mathbf{k}[x_{n+1}] \cong R(n) \otimes \mathbf{k}[t_i]. \end{aligned}$$

Similarly, there is an another map  $\varphi_2: R(n+1) \rightarrow \mathbf{k}[x_1] \otimes R(n)$  given by

$$(3.5) \quad \begin{aligned} R(n+1) \rightarrow \text{Coker}(\Phi) &\cong \frac{\bigoplus_{a=0}^n \tau_a \cdots \tau_1 R(1, n)}{\bigoplus_{a=0}^{n-1} \tau_a \cdots \tau_1 R(1, n)} \xleftarrow{\sim} \tau_n \cdots \tau_1 R(1, n) \xleftarrow{\sim} R(1, n) \\ &\cong \mathbf{k}[x_1] \otimes R(n) \cong \mathbf{k}[t_i] \otimes R(n). \end{aligned}$$

We claim that the maps  $\varphi_1$  and  $\varphi_2$  coincide with each other, which is an immediate consequence of the following lemma. When  $a_{ii} = 2$  for all  $i \in I$ , the proof easily follows from (2.3) and (2.4). However, when  $a_{ii} \neq 2$  for some  $i \in I$ , the verification becomes more complicated.

**Lemma 3.7.** *For all  $1 \leq k \leq n$  and  $1 \leq \ell \leq n-1$ ,*

- (a)  $x_k \tau_n \cdots \tau_1 \equiv \tau_n \cdots \tau_1 x_{k+1},$
- (b)  $\tau_\ell \tau_n \cdots \tau_1 \equiv \tau_n \cdots \tau_1 \tau_{\ell+1},$
- (c)  $x_{n+1} \tau_n \cdots \tau_1 \equiv \tau_n \cdots \tau_1 x_1 \pmod{R(n)R^1(n)}.$

*Proof.* We will verify that

$$(3.6) \quad \begin{aligned} &\text{for } f \in \mathbf{k}[x_1, \dots, x_{n+1}], \\ &\tau_n \tau_{n-1} \cdots \tau_k f \tau_\ell \cdots \tau_1 \equiv 0 \pmod{R(n)R^1(n)} \text{ if } \ell + 2 \leq k \leq n + 1. \end{aligned}$$

We shall prove this by using downward induction on  $k$ . If  $k = n + 1$ , it is trivial.

Assume that  $k \leq n$  and our assertion is true for  $k + 1$ . Then we have

$$(3.7) \quad \begin{aligned} \tau_n \cdots \tau_k f \tau_\ell \cdots \tau_1 &= \tau_n \cdots \tau_{k+1} (s_k(f) \tau_k + f') \tau_\ell \cdots \tau_1 \\ &= \tau_n \cdots \tau_{k+1} s_k(f) \tau_\ell \cdots \tau_1 \tau_k + \tau_n \cdots \tau_{k+1} f' \tau_\ell \cdots \tau_1 \end{aligned}$$

for some  $f' \in \mathbf{k}[x_1, \dots, x_{n+1}]$ . Since  $\tau_k \in R^1(n)$ , all the terms in the right-hand side of (3.7) are 0 mod  $R(n)R^1(n)$  by the induction hypothesis. Hence our assertion holds.

(a) For  $1 \leq k \leq n$ , we have

$$\begin{aligned} x_k \tau_n \cdots \tau_1 &= \tau_n \cdots \tau_{k+1} x_k \tau_k \cdots \tau_1 \\ &= \tau_n \cdots \tau_{k+1} \tau_k x_{k+1} \tau_{k-1} \cdots \tau_1 - \tau_n \cdots \tau_{k+1} \mathcal{P}_{k,k+1} \tau_{k-1} \cdots \tau_1. \end{aligned}$$

Then the second term is 0 mod  $R(n)R^1(n)$  by (3.6), and the first term is equal to

$$(\tau_n \cdots \tau_{k+1} \tau_k)(\tau_{k-1} \cdots \tau_1) x_{k+1},$$

which implies our first assertion.

(b) For  $1 \leq \ell \leq n-1$ , we have

$$\begin{aligned} \tau_\ell \tau_n \cdots \tau_1 &= \tau_n \cdots \tau_{\ell+2} \tau_\ell \tau_{\ell+1} \tau_\ell \cdots \tau_1 \\ &= \tau_n \cdots \tau_{\ell+2} (\tau_{\ell+1} \tau_\ell \tau_{\ell+1} - \overline{\mathcal{Q}}_\ell \mathcal{P}_{\ell,\ell+2} - \overline{\mathcal{P}}'_\ell \tau_\ell - \tau_{\ell+1} \overline{\mathcal{P}}''_\ell) \tau_{\ell-1} \cdots \tau_1 \\ &= \tau_n \cdots \tau_1 \tau_{\ell+1} - \tau_n \cdots \tau_{\ell+2} (\overline{\mathcal{Q}}_\ell \mathcal{P}_{\ell,\ell+2}) \tau_{\ell-1} \cdots \tau_1 \\ &\quad - \tau_n \cdots \tau_{\ell+2} (\overline{\mathcal{P}}'_\ell) \tau_\ell \cdots \tau_1 - \tau_n \cdots \tau_{\ell+1} (\overline{\mathcal{P}}''_\ell) \tau_{\ell-1} \cdots \tau_1. \end{aligned}$$

By (3.6), the terms except the first one are 0 mod  $R(n)R^1(n)$ .

(c) If  $k = n+1$ , we have

$$\begin{aligned} x_{n+1} \tau_n \cdots \tau_1 &= (\tau_n x_n + \mathcal{P}_{n,n+1}) \tau_{n-1} \cdots \tau_1 \\ &= \tau_n x_n \tau_{n-1} \cdots \tau_1 + \mathcal{P}_{n,n+1} \tau_{n-1} \cdots \tau_1 \\ &\equiv \tau_n x_n \tau_{n-1} \cdots \tau_1 \\ &\quad \vdots \\ &\equiv \tau_n \cdots \tau_1 x_1 \pmod{R(n)R^1(n)}. \end{aligned}$$

□

As an immediate corollary, we obtain

**Corollary 3.8.** *There is an exact sequence of  $(R(n), R(n))$ -bimodules*

$$(3.8) \quad 0 \rightarrow R(n) \otimes_{R(n-1)} R(n) \rightarrow R(n+1) \xrightarrow{\varphi} R(n) \otimes \mathbf{k}[t_i] \rightarrow 0,$$

where the map  $\varphi$  is given by (3.4) or (3.5). Here, the right  $R(n)$ -module structure on  $R(n+1)$  is given by the embedding  $\xi_n: R(n) \xrightarrow{\sim} R^1(n) \hookrightarrow R(n+1)$ . Moreover, both the

left multiplication by  $x_{n+1}$  and the right multiplication by  $x_1$  on  $R(n+1)$  are compatible with the multiplication by  $t_i$  on  $R(n) \otimes \mathbf{k}[t_i]$ .

By applying the exact functor  $e(\beta + \alpha_j - \alpha_i, i) \bullet e(j, \beta)$  on (3.8), Corollary 3.8 yields the following theorem.

**Theorem 3.9.**

(i) *There is a natural isomorphism*

$$\overline{F}_j E_i \xrightarrow{\sim} E_i \overline{F}_j \quad \text{for } i \neq j.$$

(ii) *There is an exact sequence in  $\text{Mod}(R(\beta))$ :*

$$0 \rightarrow \overline{F}_i E_i M \rightarrow E_i \overline{F}_i M \rightarrow q^{-(\alpha_i|\beta)} M \otimes \mathbf{k}[t_i] \rightarrow 0,$$

*which is functorial in  $M$ .*

#### 4. THE CYCLOTOMIC QUOTIENT $R^\Lambda$

In this section, we define the cyclotomic Khovanov-Lauda-Rouquier algebra  $R^\Lambda$  and the functors  $E_i^\Lambda, F_i^\Lambda$  on  $\text{Mod}(R^\Lambda)$ . We investigate the structure of  $R^\Lambda$  and the behavior of  $E_i^\Lambda, F_i^\Lambda$  on  $\text{Proj}(R^\Lambda)$  and  $\text{Rep}(R^\Lambda)$ . In particular, we will show that  $E_i^\Lambda$  and  $F_i^\Lambda$  are well-defined exact functors on  $\text{Proj}(R^\Lambda)$  and  $\text{Rep}(R^\Lambda)$ .

For  $\Lambda \in \mathbf{P}^+$  and  $i \in I$ , we choose a monic polynomial of degree  $\langle h_i, \Lambda \rangle$

$$(4.1) \quad a_i^\Lambda(u) = \sum_{k=0}^{\langle h_i, \Lambda \rangle} c_{i;k} u^{\langle h_i, \Lambda \rangle - k}$$

with  $c_{i;k} \in \mathbf{k}_{2kd_i}$  and  $c_{i,0} = 1$ .

Given  $\beta \in \mathbf{Q}^+$  with  $|\beta| = n$ , a dominant integral weight  $\Lambda \in \mathbf{P}^+$  and  $k$  ( $1 \leq k \leq n$ ), set

$$a^\Lambda(x_k) = \sum_{\nu \in I^\beta} a_{\nu_k}^\Lambda(x_k) e(\nu) \in R(\beta).$$

**Definition 4.1.** Let  $\beta \in \mathbf{Q}^+$  and  $\Lambda \in \mathbf{P}^+$ .

(1) The cyclotomic Khovanov-Lauda-Rouquier algebra  $R^\Lambda(\beta)$  at  $\beta$  is the quotient algebra

$$R^\Lambda(\beta) = \frac{R(\beta)}{R(\beta) a^\Lambda(x_1) R(\beta)}.$$

- (2) The  $\mathbb{Q}^+$ -graded algebra  $R^\Lambda = \bigoplus_{\alpha \in \mathbb{Q}^+} R^\Lambda(\alpha)$  is called the *cyclotomic Khovanov-Lauda-Rouquier algebra* of weight  $\Lambda$ .

**Lemma 4.2.** *Let  $\nu \in I^n$  be such that  $\nu_a = \nu_{a+1}$  for some  $1 \leq a < n$ . Then, for an  $R(n)$ -module  $M$  and  $f \in \mathbf{k}[x_1, \dots, x_n]$ ,  $fe(\nu)M = 0$  implies*

$$\begin{aligned} (\partial_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \mathcal{P}_{\nu_a}(x_{a+1}, x_a) e(\nu) M &= 0, \\ (s_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \mathcal{P}_{\nu_a}(x_{a+1}, x_a) e(\nu) M &= 0. \end{aligned}$$

*Proof.* Note that  $\tau_a e(\nu) = e(\nu) \tau_a$  and  $\tau_a^2 e(\nu) = (\partial_a \mathcal{P}_{\nu_a}(x_a, x_{a+1})) \tau_a e(\nu)$ . Thus we have

$$\begin{aligned} & (x_a - x_{a+1}) \tau_a f \tau_a e(\nu) \\ &= (x_a - x_{a+1}) ((s_a f) \tau_a + (\partial_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1})) \tau_a e(\nu) \\ &= (x_a - x_{a+1}) \left( (\partial_a \mathcal{P}_{\nu_a}(x_a, x_{a+1})) (s_a f) + (\partial_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \right) \tau_a e(\nu) \\ &= (\mathcal{P}_{\nu_a}(x_{a+1}, x_a) - \mathcal{P}_{\nu_a}(x_a, x_{a+1})) (s_a f) \tau_a e(\nu) + \mathcal{P}_{\nu_a}(x_a, x_{a+1}) (s_a(f) - f) \tau_a e(\nu) \\ &= \mathcal{P}_{\nu_a}(x_{a+1}, x_a) (s_a f) \tau_a e(\nu) - \mathcal{P}_{\nu_a}(x_a, x_{a+1}) f \tau_a e(\nu) \\ &= \mathcal{P}_{\nu_a}(x_{a+1}, x_a) (\tau_a f - (\partial_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1})) e(\nu) - \mathcal{P}_{\nu_a}(x_a, x_{a+1}) f \tau_a e(\nu). \end{aligned}$$

Thus

$$(\partial_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \mathcal{P}_{\nu_a}(x_{a+1}, x_a) e(\nu) M = 0.$$

Since  $(x_a - x_{a+1})(\partial_a f) = s_a f - f$ , we have

$$(s_a f) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \mathcal{P}_{\nu_a}(x_{a+1}, x_a) e(\nu) M = 0.$$

□

**Lemma 4.3.** *Let  $\beta \in \mathbb{Q}^+$  with  $|\beta| = n$ .*

- (i) *There exists a monic polynomial  $g(u)$  such that  $g(x_a) = 0$  in  $R^\Lambda(\beta)$  for any  $a$  ( $1 \leq a \leq n$ ).*
- (ii) *If  $i \in I^{\text{re}}$ , then there exists  $m \in \mathbb{Z}_{\geq 0}$  such that  $R^\Lambda(\beta + k\alpha_i) = 0$  for any  $k \geq m$ .*

*Proof.* (i) By induction on  $a$ , it is enough to show that

For any monic polynomial  $g(u)$ , we can find a monic polynomial  $h(u)$  such that we have  $h(x_{a+1})M = 0$  for any  $R(\beta)$ -module  $M$  with  $g(x_a)M = 0$ .

If  $\nu_a = \nu_{a+1}$ , then Lemma 4.2 implies that

$$g(x_{a+1}) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \mathcal{P}_{\nu_a}(x_{a+1}, x_a) e(\nu) M = 0.$$



By the definition of  $\mathcal{P}_i(u, v)$  given in (2.1),  $(g(x_{a+1})\mathcal{P}_{\nu_a}(x_a, x_{a+1})\mathcal{P}_{\nu_a}(x_{a+1}, x_a))$  is a monic polynomial in  $x_{a+1}$  with coefficient in  $\mathbf{k}[x_a]$ . Hence we can choose a monic polynomial  $h(x_{a+1})$  in the ideal generated by  $g(x_a)$  and  $g(x_{a+1})\mathcal{P}_{\nu_a}(x_a, x_{a+1})\mathcal{P}_{\nu_a}(x_{a+1}, x_a)$  in  $\mathbf{k}[x_a, x_{a+1}]$ . Thus

$$h(x_{a+1})e(\nu)M = 0.$$

If  $\nu_a \neq \nu_{a+1}$ , then

$$g(x_{a+1})\mathcal{Q}_{\nu_a, \nu_{a+1}}(x_a, x_{a+1})e(\nu)M = g(x_{a+1})\tau_a^2 e(\nu)M = \tau_a g(x_a)e(s_a \nu)\tau_a M = 0.$$

Since  $g(x_{a+1})\mathcal{Q}_{\nu_a, \nu_{a+1}}(x_a, x_{a+1})$  is a monic polynomial in  $x_{a+1}$  with coefficient in  $\mathbf{k}[x_a]$ , we can choose a monic polynomial  $h(x_{a+1})$  as in the case of  $\nu_a = \nu_{a+1}$ .

(ii) For  $\nu \in I^n$ , set  $\text{Supp}_i(\nu) = \#\{k \mid 1 \leq k \leq n \text{ and } \nu_k = i\}$ . Our assertion is equivalent to:

$$(4.2) \quad \begin{aligned} &\text{For all } n, \text{ there exists } k_n \in \mathbb{Z}_{\geq 0} \text{ such that } e(\nu)R^\Lambda(n + k_n) = 0 \\ &\text{for any } \nu \in I^{n+k_n} \text{ with } \text{Supp}_i(\nu) \geq k_n. \end{aligned}$$

If  $e(\nu)R^\Lambda(n + k) = 0$  for any  $\nu \in I^{n+k}$  such that  $\text{Supp}_i(\nu) \geq k$ , then one can easily see that

$$(4.3) \quad e(\nu')R^\Lambda(n + k') = 0 \text{ for any } k' \geq k \text{ and } \nu' \in I^{n+k'} \text{ such that } \text{Supp}_i(\nu'_{\leq n+k}) \geq k.$$

In order to prove (4.2), we will use induction on  $n$ . Assume that there exists  $k = k_{n-1}$  such that

$$e(\nu)R^\Lambda(n - 1 + k) = 0 \quad \text{if } \text{Supp}_i(\nu) \geq k.$$

By (i), there exists a monic polynomial  $g(u)$  of degree  $m \geq 0$  such that  $g(x_{n+k})R^\Lambda(n + k) = 0$ . It suffices to show

$$e(\nu)R^\Lambda(n + k + m) = 0 \text{ for } \text{Supp}_i(\nu) \geq k + m.$$

If  $\text{Supp}_i(\nu_{\leq n+k-1}) \geq k$ , then by (4.3)  $e(\nu)R^\Lambda(n + k + m) = 0$ . Thus we may assume that  $\text{Supp}_i(\nu_{\leq n+k-1}) \leq k - 1$ . Hence we have  $\nu_{\geq n+k} = (i, \dots, i)$ . Then the repeated application of Lemma 4.2 implies

$$(\partial_{n+k+m-1} \cdots \partial_{n+k} g(x_{n+k}))e(\nu)R^\Lambda(n + k + m) = 0.$$

Since  $\partial_{n+k+m-1} \cdots \partial_{n+k} g(x_{n+k}) = \pm 1$ , we can choose  $k_n = k + m$ . □

**Lemma 4.4.** *If  $i \in I^{\text{im}}$  and  $\langle h_i, \Lambda - \beta \rangle = 0$ , then*

$$R^\Lambda(\beta + \alpha_i) = 0.$$

*Proof.* Since  $\langle h_i, \Lambda \rangle, \langle h_i, -\beta \rangle \geq 0$ , the hypothesis  $\langle h_i, \Lambda - \beta \rangle = 0$  implies  $\langle h_i, \Lambda \rangle = 0$  and  $\langle h_i, \beta \rangle = 0$ . Thus for all  $j \in \text{Supp}(\beta) \setminus \{i\}$ , we have  $a_{ij} = 0$ . In particular we have  $\mathcal{Q}_{j,i} \in \mathbf{k}_0^\times$ . Since  $\langle h_i, \Lambda \rangle = 0$ , we have  $e(i, \beta)R^\Lambda(\beta + \alpha_i) = 0$ . For  $\nu \in I^{\beta+\alpha_i}$ , let  $k$  be the smallest integer such that  $\nu_k = i$ . We shall show  $e(\nu)R^\Lambda(\beta + \alpha_i) = 0$  by induction on  $k$ . If  $k=1$ , it is obvious. Assume  $k > 1$ . Hence  $\mathcal{Q}_{\nu_{k-1}, \nu_k} e(\nu)R^\Lambda(\beta + \alpha_i) = \tau_{k-1}e(s_{k-1}\nu)\tau_{k-1}R^\Lambda(\beta + \alpha_i)$  vanishes since  $(s_{k-1}\nu)_{k-1} = i$ . Since  $\mathcal{Q}_{\nu_{k-1}, \nu_k} \in \mathbf{k}_0^\times$ , we obtain the desired result  $e(\nu)R^\Lambda(\beta + \alpha_i) = 0$ .  $\square$

For each  $i \in I$ , we define the functors

$$\begin{aligned} E_i^\Lambda &: \text{Mod}(R^\Lambda(\beta + \alpha_i)) \rightarrow \text{Mod}(R^\Lambda(\beta)), \\ F_i^\Lambda &: \text{Mod}(R^\Lambda(\beta)) \rightarrow \text{Mod}(R^\Lambda(\beta + \alpha_i)), \end{aligned}$$

by

$$\begin{aligned} E_i^\Lambda(N) &= e(\beta, i)N = e(\beta, i)R^\Lambda(\beta + \alpha_i) \otimes_{R^\Lambda(\beta + \alpha_i)} N, \\ F_i^\Lambda(M) &= R^\Lambda(\beta + \alpha_i)e(\beta, i) \otimes_{R^\Lambda(\beta)} M, \end{aligned}$$

where  $M \in \text{Mod}(R^\Lambda(\beta + \alpha_i))$  and  $N \in \text{Mod}(R^\Lambda(\beta))$ .

We introduce  $(R(\beta + \alpha_i), R^\Lambda(\beta))$ -bimodules

$$\begin{aligned} F^\Lambda &= R^\Lambda(\beta + \alpha_i)e(\beta, i) = \frac{R(\beta + \alpha_i)e(\beta, i)}{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta + \alpha_i)e(\beta, i)}, \\ (4.4) \quad K_0 &= R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} R^\Lambda(\beta) = \frac{R(\beta + \alpha_i)e(\beta, i)}{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)}, \\ K_1 &= R(\beta + \alpha_i)e(i, \beta) \otimes_{R(\beta)} R^\Lambda(\beta) = \frac{R(\beta + \alpha_i)e(i, \beta)}{R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \beta)}. \end{aligned}$$

The right  $R(\beta)$ -module structure on  $R(\beta + \alpha_i)e(i, \beta)$  and the right  $R^\Lambda(\beta)$ -module structure on  $K_1$  are given by the isomorphism  $R(\beta) \xrightarrow{\sim} R^1(\beta) \hookrightarrow R(\beta + \alpha_i)$ . The bimodules  $F^\Lambda$ ,  $K_0$  and  $K_1$  are the kernels of the functors  $F_i^\Lambda$ ,  $F_i$  and  $\overline{F}_i$  from  $\text{Mod}(R^\Lambda(\beta))$  to  $\text{Mod}(R(\beta + \alpha_i))$ , respectively.

Let  $t_i$  be an indeterminate of degree  $2d_i$ . Then  $\mathbf{k}[t_i]$  acts from the right on  $R(\beta + \alpha_i)e(i, \beta)$  and  $K_1$  by multiplying  $x_1$ . Similarly,  $\mathbf{k}[t_i]$  acts from the right on  $R(\beta + \alpha_i)e(\beta, i)$ ,  $F^\Lambda$  and  $K_1$  by multiplying  $x_{n+1}$ . Thus  $K_0$ ,  $F^\Lambda$  and  $K_1$  have an  $(R(\beta + \alpha_i), R^\Lambda(\beta) \otimes \mathbf{k}[t_i])$ -bimodule structure.

By a similar argument to the one given in [7, Lemma 4.8, Lemma 4.16], we obtain the following lemmas which will be used in proving Corollary 4.12 and Theorem 4.13.

**Lemma 4.5.**

- (i) Both  $K_1$  and  $K_0$  are finitely generated projective right  $R^\Lambda(\beta) \otimes \mathbf{k}[t_i]$ -modules.
- (ii) In particular, for any  $f(x_1, \dots, x_{n+1}) \in \mathbf{k}[x_1, \dots, x_{n+1}]$  which is a monic polynomial in  $x_1$ , the right multiplication by  $f$  on  $K_1$  induces an injective endomorphism of  $K_1$ .

**Lemma 4.6.** For  $i \in I$  and  $\beta \in \mathbb{Q}^+$  with  $|\beta| = n$ , we have

- (i)  $R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta + \alpha_i) = \sum_{a=0}^n R(\beta + \alpha_i)a^\Lambda(x_1)\tau_1 \cdots \tau_a$ ,
- (ii)  $R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta + \alpha_i)e(\beta, i)$   
 $= R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i) + R(\beta + \alpha_i)a^\Lambda(x_1)\tau_1 \cdots \tau_n e(\beta, i).$

Let  $\pi: K_0 \rightarrow F^\Lambda$  be the canonical projection and  $\tilde{P}: R(\beta + \alpha_i)e(i, \beta) \rightarrow K_0$  be the right multiplication by  $a^\Lambda(x_1)\tau_1 \cdots \tau_n$  whose degree is

$$2d_i\langle h_i, \Lambda \rangle + (\alpha_i| - \beta) = (\alpha_i|2\Lambda - \beta).$$

Then, using Lemma 4.6, one can see that

$$(4.5) \quad \text{Im}(\tilde{P}) = \text{Ker}\pi = \frac{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta + \alpha_i)e(\beta, i)}{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)} \subset K_0.$$

**Lemma 4.7.** The map  $\tilde{P}: R(\beta + \alpha_i)e(i, \beta) \rightarrow K_0$  is a right  $R(\beta) \otimes \mathbf{k}[t_i]$ -linear homomorphism; i.e., for all  $S \in R(\beta + \alpha_i)$ ,  $1 \leq a \leq n$  and  $1 \leq b \leq n-1$ ,

$$\tilde{P}(Sx_{a+1}) = \tilde{P}(S)x_a, \quad \tilde{P}(Sx_1) = \tilde{P}(S)x_{n+1}, \quad \tilde{P}(S\tau_{b+1}) = \tilde{P}(S)\tau_b.$$

*Proof.* First, we will verify that

$$(4.6) \quad \text{for any } f \in \mathbf{k}[x_1, \dots, x_{n+1}], \quad a^\Lambda(x_1)\tau_1 \cdots \tau_\ell f \tau_k \cdots \tau_n e(\beta, i) \equiv 0 \pmod{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)} \text{ if } \ell + 2 \leq k \leq n + 1.$$

We will prove this by using downward induction on  $k$ . It is trivial for  $k = n + 1$ . Assume that  $k \leq n$  and our assertion is true for  $k + 1$ . Then we have

$$(4.7) \quad \begin{aligned} a^\Lambda(x_1)\tau_1 \cdots \tau_\ell f \tau_k \cdots \tau_n e(\beta, i) &= \tau_k a^\Lambda(x_1)\tau_1 \cdots \tau_\ell s_k(f) \tau_{k+1} \cdots \tau_n e(\beta, i) \\ &\quad + a^\Lambda(x_1)\tau_1 \cdots \tau_\ell f' \tau_{k+1} \cdots \tau_n e(\beta, i) \end{aligned}$$

for some  $f' \in \mathbf{k}[x_1, \dots, x_{n+1}]$ , and both the terms in the right-hand side of (4.7) are 0 mod  $R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)$  by the induction hypothesis. Thus we obtain (4.6).

For  $1 \leq a \leq n$ , we have

$$\begin{aligned}
x_{a+1}(a^\Lambda(x_1)\tau_1 \cdots \tau_n e(\beta, i)) &= a^\Lambda(x_1)\tau_1 \cdots \tau_{a-1}(x_{a+1}\tau_a)\tau_{a+1} \cdots \tau_n e(\beta, i), \\
&= a^\Lambda(x_1)\tau_1 \cdots \tau_{a-1}(\tau_a x_a + \mathcal{P}_{a,a+1})\tau_{a+1} \cdots \tau_n e(\beta, i), \\
&= a^\Lambda(x_1)\tau_1 \cdots \tau_n x_a e(\beta, i) + a^\Lambda(x_1)\tau_1 \cdots \tau_{a-1}\mathcal{P}_{a,a+1}\tau_{a+1} \cdots \tau_n e(\beta, i), \\
&\equiv a^\Lambda(x_1)\tau_1 \cdots \tau_n x_a e(\beta, i) \quad (\text{by (4.6)}).
\end{aligned}$$

For the second assertion, we have

$$\begin{aligned}
x_1(a^\Lambda(x_1)\tau_1 \cdots \tau_n e(\beta, i)) &= a^\Lambda(x_1)(\tau_1 x_2 - \mathcal{P}_{1,2})\tau_2 \cdots \tau_n e(\beta, i) \\
&= a^\Lambda(x_1)\tau_1 x_2 \tau_2 \cdots \tau_n e(\beta, i) - \mathcal{P}_{1,2}\tau_2 \cdots \tau_n a^\Lambda(x_1)e(\beta, i) \\
&\equiv a^\Lambda(x_1)\tau_1 x_2 \tau_2 \cdots \tau_n e(\beta, i) \\
&= a^\Lambda(x_1)\tau_1 \tau_2 x_3 \tau_3 \cdots \tau_n e(\beta, i) - a^\Lambda(x_1)\tau_1 \mathcal{P}_{2,3}\tau_3 \cdots \tau_n e(\beta, i) \\
&\equiv a^\Lambda(x_1)\tau_1 \tau_2 x_3 \tau_3 \cdots \tau_n e(\beta, i) \quad (\text{by (4.6)}) \\
&\vdots \\
&\equiv a^\Lambda(x_1)\tau_1 \cdots \tau_n x_{n+1} e(\beta, i) \mod R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i).
\end{aligned}$$

For  $1 \leq b \leq n-1$ , we have

$$\begin{aligned}
&\tau_{b+1}(a^\Lambda(x_1)\tau_1 \cdots \tau_n e(\beta, i)) \\
&= a^\Lambda(x_1)\tau_1 \cdots \tau_{b-1}(\tau_{b+1}\tau_b\tau_{b+1})\tau_{b+2} \cdots \tau_n e(\beta, i) \\
&= a^\Lambda(x_1)\tau_1 \cdots \tau_{b-1}(\tau_b\tau_{b+1}\tau_b + \overline{\mathcal{Q}}_b\mathcal{P}_{b,b+2} + \tau_b\overline{\mathcal{P}}'_b + \overline{\mathcal{P}}''_b\tau_{b+1})\tau_{b+2} \cdots \tau_n e(\beta, i) \\
&= a^\Lambda(x_1)\tau_1 \cdots \tau_n \tau_b e(\beta, i) + a^\Lambda(x_1)\tau_1 \cdots \tau_{b-1}(\overline{\mathcal{Q}}_b\mathcal{P}_{b,b+2})\tau_{b+2} \cdots \tau_n e(\beta, i) \\
&\quad + a^\Lambda(x_1)\tau_1 \cdots \tau_b(\overline{\mathcal{P}}'_b)\tau_{b+2} \cdots \tau_n e(\beta, i) + a^\Lambda(x_1)\tau_1 \cdots \tau_{b-1}(\overline{\mathcal{P}}''_b)\tau_{b+1} \cdots \tau_n e(\beta, i).
\end{aligned}$$

By (4.6), all the terms except the first one are 0 mod  $R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)$ . Thus we obtain

$$\tau_{b+1}a^\Lambda(x_1)\tau_1 \cdots \tau_n e(\beta, i) \equiv a^\Lambda(x_1)\tau_1 \cdots \tau_n \tau_b e(\beta, i) \mod R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i).$$

□

Since  $\tilde{P}$  is right  $R(\beta) \otimes \mathbf{k}[t_i]$ -linear and maps  $R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \beta)$  to  $R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)$ , it induces a map

$$P: K_1 \rightarrow K_0,$$

which is an  $(R(\beta + \alpha_i), R(\beta) \otimes \mathbf{k}[t_i])$ -bilinear homomorphism. By (4.5), we get an exact sequence of  $(R(\beta + \alpha_i), R(\beta) \otimes \mathbf{k}[t_i])$ -bimodules

$$K_1 \xrightarrow{P} K_0 \xrightarrow{\pi} F^\Lambda \longrightarrow 0.$$

We will show that  $P$  is actually injective by constructing an  $(R(\beta + \alpha_i), R(\beta) \otimes \mathbf{k}[t_i])$ -bilinear homomorphism  $Q$  such that  $Q \circ P$  is injective.

For  $1 \leq a \leq n$ , we define an element  $g_a$  of  $R(\beta + \alpha_i)$  by

$$(4.8) \quad g_a = \sum_{\substack{\nu \in I^{\beta + \alpha_i}, \\ \nu_a \neq \nu_{a+1}}} \tau_a e(\nu) + \sum_{\substack{\nu \in I^{\beta + \alpha_i}, \\ \nu_a = \nu_{a+1}}} ((x_{a+1} - x_a) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) - (x_{a+1} - x_a)^2 \tau_a) e(\nu).$$

**Lemma 4.8.** *For  $1 \leq a \leq n$ , we have*

$$(4.9) \quad x_{s_a(b)} g_a = g_a x_b \quad (1 \leq b \leq n+1) \quad \text{and} \quad \tau_a g_{a+1} g_a = g_{a+1} g_a \tau_{a+1}.$$

*Proof.* For  $\nu$  such that  $\nu_a \neq \nu_{a+1}$ , we have

$$(4.10) \quad x_{s_a(b)} g_a e(\nu) = g_a x_b e(\nu).$$

We shall show (4.10) when  $\nu_a = \nu_{a+1}$ . We have

$$\begin{aligned} & (x_a g_a - g_a x_{a+1}) e(\nu) \\ &= \{x_a (x_{a+1} - x_a) \mathcal{P}_{\nu_a}(x_a, x_{a+1}) - x_a (x_{a+1} - x_a)^2 \tau_a - (x_{a+1} - x_a) x_{a+1} \mathcal{P}_{\nu_a}(x_a, x_{a+1}) \\ & \quad + (x_{a+1} - x_a)^2 (x_a \tau_a + \mathcal{P}_{\nu_a}(x_a, x_{a+1})) e(\nu)\} \\ &= \{-(x_{a+1} - x_a)^2 \mathcal{P}_{\nu_a}(x_a, x_{a+1}) + (x_{a+1} - x_a)^2 \mathcal{P}_{\nu_a}(x_a, x_{a+1})\} e(\nu) = 0. \end{aligned}$$

Hence (4.10) holds for  $b = a+1$ . The other cases can be proved similarly.

By (2.4),  $S = \tau_a g_{a+1} g_a - g_{a+1} g_a \tau_{a+1}$  does not contain the term  $\tau_{a+1} \tau_a \tau_{a+1}$  and  $\tau_a \tau_{a+1} \tau_a$  and is contained in the  $\mathbf{k}[x_a, x_{a+1}, x_{a+2}]$ -module generated by  $1, \tau_a, \tau_{a+1}, \tau_a \tau_{a+1}, \tau_{a+1} \tau_a$ . That is,  $S$  can be expressed as

$$S = \mathsf{T}_1 + \mathsf{T}_2 \tau_a + \mathsf{T}_3 \tau_{a+1} + \mathsf{T}_4 \tau_a \tau_{a+1} + \mathsf{T}_5 \tau_{a+1} \tau_a$$

for some  $\mathsf{T}_i \in \mathbf{k}[x_a, x_{a+1}, x_{a+2}]$  ( $1 \leq i \leq 5$ ). By a similar argument given in [7, Lemma 4.12], we have

$$S x_b = x_{s_{a,a+2}(b)} S \quad \text{for all } b.$$

Then one can show that all  $\mathsf{T}_i$  must be zero. Thus our second assertion holds.  $\square$

**Proposition 4.9.**

- (i) Let  $\tilde{Q}: R(\beta + \alpha_i)e(\beta, i) \rightarrow K_1$  be the left  $R(\beta + \alpha_i)$ -linear homomorphism given by the multiplication of  $g_n \cdots g_1$  from the right. Then  $\tilde{Q}$  is a right  $(R(\beta) \otimes k[t_i])$ -linear homomorphism. That is,

$$\begin{aligned}\tilde{Q}(Sx_a) &= \tilde{Q}(S)x_{a+1} \quad (1 \leq a \leq n), \quad \tilde{Q}(Sx_{n+1}) = \tilde{Q}(S)x_1 \\ \tilde{Q}(S\tau_b) &= \tilde{Q}(S)\tau_{b+1} \quad (1 \leq b \leq n-1)\end{aligned}$$

for any  $S \in R(\beta + \alpha_i)e(\beta, i)$ .

- (ii) The map  $\tilde{Q}$  induces a well-defined  $(R(\beta + \alpha_i), R(\beta) \otimes k[t_i])$ -bilinear homomorphism

$$Q: K_0 = \frac{R(\beta + \alpha_i)e(\beta, i)}{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\beta, i)} \rightarrow K_1 = \frac{R(\beta + \alpha_i)e(i, \beta)}{R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \beta)}.$$

*Proof.* The proof follows immediately from the preceding lemma.  $\square$

**Theorem 4.10.** For each  $\nu \in I^\beta$ , set

$$A_\nu = a_i^\Lambda(x_1) \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} Q_{i, \nu_a}(x_1, x_{a+1}) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(x_1, x_{a+1}) \mathcal{P}_i(x_{a+1}, x_1).$$

Then the following diagram is commutative, in which the vertical arrow is the multiplication by  $A_\nu$  from the right.

$$(4.11) \quad \begin{array}{ccc} \frac{R(\beta + \alpha_i)e(i, \nu)}{R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \nu)} & \xrightarrow{P=a^\Lambda(x_1)\tau_1 \cdots \tau_n} & \frac{R(\beta + \alpha_i)e(\nu, i)}{R(\beta + \alpha_i)a^\Lambda(x_1)R(\beta)e(\nu, i)} \\ A_\nu \downarrow & \swarrow Q=g_n \cdots g_1 & \\ \frac{R(\beta + \alpha_i)e(i, \nu)}{R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \nu)} & & \end{array}$$

*Proof.* It suffices to show that

$$(4.12) \quad \begin{aligned} a^\Lambda(x_1)\tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) &= a^\Lambda(x_1)\tau_1 \cdots \tau_n e(\nu, i) g_n \cdots g_1 \\ &\equiv A_\nu e(i, \nu) \pmod{R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \nu)}. \end{aligned}$$

Note that

$$(4.13) \quad \tau_n e(\nu, i) g_n = \begin{cases} \tau_n e(\nu, i) \tau_n = Q_{i, \nu_n}(x_n, x_{n+1}) e(\nu_{< n}, i, \nu_n) & \text{if } \nu_n \neq i, \\ \tau_n (x_{n+1} - x_n) \mathcal{P}_i(x_{n+1}, x_n) e(\nu, i) & \text{if } \nu_n = i. \end{cases}$$

Indeed if  $\nu_n = i$ , then we have

$$\begin{aligned}
& \tau_n \left( (x_{n+1} - x_n) \mathcal{P}_i(x_n, x_{n+1}) - (x_{n+1} - x_n)^2 \tau_n \right) e(\nu, i) \\
&= \left( \tau_n (x_{n+1} - x_n) \mathcal{P}_i(x_n, x_{n+1}) - \tau_n^2 (x_{n+1} - x_n)^2 \right) e(\nu, i) \\
&= \left( \tau_n (x_{n+1} - x_n) \mathcal{P}_i(x_n, x_{n+1}) - \tau_n (\partial_n \mathcal{P}_i(x_n, x_{n+1})) (x_{n+1} - x_n)^2 \right) e(\nu, i) \\
&= \tau_n (x_{n+1} - x_n) \left( \mathcal{P}_i(x_n, x_{n+1}) - (\mathcal{P}_i(x_n, x_{n+1}) - \mathcal{P}_i(x_{n+1}, x_n)) \right) e(\nu, i) \\
&= \tau_n (x_{n+1} - x_n) \mathcal{P}_i(x_{n+1}, x_n) e(\nu, i).
\end{aligned}$$

We will show (4.12) by induction on  $n$ . Assume first  $n = 1$ . If  $\nu_1 \neq i$ , then it is already given by (4.13). If  $\nu_1 = i$ , then

$$\begin{aligned}
a^\Lambda(x_1) \tau_1 e(i, i) g_1 &= a^\Lambda(x_1) \tau_1 (x_2 - x_1) \mathcal{P}_i(x_2, x_1) e(i, i) \\
&= \left( \tau_1 a^\Lambda(x_2) + \frac{a^\Lambda(x_2) - a^\Lambda(x_1)}{x_1 - x_2} \mathcal{P}_i(x_1, x_2) \right) (x_2 - x_1) \mathcal{P}_i(x_2, x_1) e(i, i) \\
&= \left( \tau_1 a^\Lambda(x_2) (x_2 - x_1) \mathcal{P}_i(x_2, x_1) - (a^\Lambda(x_2) - a^\Lambda(x_1)) \mathcal{P}_i(x_1, x_2) \mathcal{P}_i(x_2, x_1) \right) e(i, i) \\
&\equiv a^\Lambda(x_1) \mathcal{P}_i(x_1, x_2) \mathcal{P}_i(x_2, x_1) e(i, i) = \mathbf{A}_\nu e(i, i).
\end{aligned}$$

Thus we may assume that  $n > 1$ .

(i) First assume that  $\nu_n \neq i$ . Then we have

$$\begin{aligned}
& a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) \\
&= a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} \mathcal{Q}_{i, \nu_n}(x_n, x_{n+1}) g_{n-1} \cdots g_1 e(i, \nu) \\
&= a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} g_{n-1} \cdots g_1 \mathcal{Q}_{i, \nu_n}(x_1, x_{n+1}) e(i, \nu) \\
&\equiv \mathbf{A}_{\nu_{< n}} \mathcal{Q}_{i, \nu_n}(x_1, x_{n+1}) e(i, \nu) = \mathbf{A}_\nu e(i, \nu).
\end{aligned}$$

(ii) If  $\nu_n = i$ , then we have

$$\begin{aligned}
(4.14) \quad & a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) \\
&= a^\Lambda(x_1) \tau_1 \cdots \tau_n (x_{n+1} - x_n) \mathcal{P}_i(x_{n+1}, x_n) g_{n-1} \cdots g_1 e(i, \nu) \\
&= a^\Lambda(x_1) \tau_1 \cdots \tau_n g_{n-1} \cdots g_1 (x_{n+1} - x_1) \mathcal{P}_i(x_{n+1}, x_1) e(i, \nu).
\end{aligned}$$

Since  $P$  and  $Q$  are right  $R(\beta) \otimes \mathbf{k}[t_i]$ -linear, we have

$$\begin{aligned}
(4.15) \quad & x_{n+1} a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) - a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 x_{n+1} e(i, \nu) \equiv 0 \\
& \text{mod } R(\beta + \alpha_i) a^\Lambda(x_2) R^1(\beta) e(i, \beta).
\end{aligned}$$

By (4.14), the left-hand side of (4.15) is equal to

$$(4.16) \quad a^\Lambda(x_1)\tau_1 \cdots \tau_{n-1}(x_{n+1}\tau_n - \tau_n x_{n+1})g_{n-1} \cdots g_1(x_{n+1} - x_1)\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu).$$

Since

$$\begin{aligned} (x_{n+1}\tau_n - \tau_n x_{n+1})e(\nu, i) &= \{(x_{n+1}\tau_n - \tau_n x_n) + \tau_n(x_n - x_{n+1})\}e(i, \nu) \\ &= \mathcal{P}_i(x_n, x_{n+1}) + \tau_n(x_n - x_{n+1}), \end{aligned}$$

we have

$$\begin{aligned} 0 &\equiv a^\Lambda(x_1)\tau_1 \cdots \tau_{n-1}\mathcal{P}_i(x_n, x_{n+1})g_{n-1} \cdots g_1(x_{n+1} - x_1)\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ &\quad + a^\Lambda(x_1)\tau_1 \cdots \tau_n(x_n - x_{n+1})g_{n-1} \cdots g_1(x_{n+1} - x_1)\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ &= a^\Lambda(x_1)\tau_1 \cdots \tau_{n-1}g_{n-1} \cdots g_1(x_{n+1} - x_1)\mathcal{P}_i(x_1, x_{n+1})\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ &\quad - a^\Lambda(x_1)\tau_1 \cdots \tau_n g_{n-1} \cdots g_1(x_{n+1} - x_1)^2\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ (4.17) \quad &\equiv \mathbf{A}_{\nu < n}(x_{n+1} - x_1)\mathcal{P}_i(x_1, x_{n+1})\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ &\quad - a^\Lambda(x_1)\tau_1 \cdots \tau_n g_{n-1} \cdots g_1(x_{n+1} - x_1)^2\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ &= \left( \mathbf{A}_{\nu < n}\mathcal{P}_i(x_1, x_{n+1}) - a^\Lambda(x_1)\tau_1 \cdots \tau_n g_{n-1} \cdots g_1(x_{n+1} - x_1) \right) e(i, \nu) \\ &\quad \times (x_{n+1} - x_1)\mathcal{P}_i(x_{n+1}, x_1). \end{aligned}$$

Since the right multiplication of  $(x_{n+1} - x_1)\mathcal{P}_i(x_{n+1}, x_1)$  on  $K_1$  is injective by Lemma 4.5, we conclude that

$$a^\Lambda(x_1)\tau_1 \cdots \tau_n g_{n-1} \cdots g_1(x_{n+1} - x_1)e(i, \nu) \equiv \mathbf{A}_{\nu < n}\mathcal{P}_i(x_1, x_{n+1})e(i, \nu).$$

Hence (4.14) implies that

$$\begin{aligned} a^\Lambda(x_1)\tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) &\equiv \mathbf{A}_{\nu < n}\mathcal{P}_i(x_1, x_{n+1})\mathcal{P}_i(x_{n+1}, x_1)e(i, \nu) \\ &= \mathbf{A}_\nu e(i, \nu) \mod R(\beta + \alpha_i)a^\Lambda(x_2)R^1(\beta)e(i, \beta). \end{aligned}$$

□

Since  $K_1 e(i, \nu)$  is a projective  $R^\Lambda(\beta) \otimes \mathbf{k}[t_i]$ -module by Lemma 4.5 and  $\mathbf{A}_\nu$  is a monic polynomial (up to a multiple of an invertible element) in  $t_i$ , by a similar argument to the one in [7, Lemma 4.17, Lemma 4.18], we conclude:

**Theorem 4.11.** *We have a short exact sequence consisting of right projective  $R^\Lambda(\beta)$ -modules:*

$$(4.18) \quad 0 \rightarrow K_1 \xrightarrow{P} K_0 \rightarrow F^\Lambda \rightarrow 0.$$



Since  $K_1$ ,  $K_0$  and  $F^\Lambda$  are kernels of functors  $\overline{F}_i$ ,  $F_i$  and  $F_i^\Lambda$ , respectively, we have

**Corollary 4.12.** *For any  $i \in I$  and  $\beta \in \mathbb{Q}^+$ , there exists an exact sequence of  $R(\beta + \alpha_i)$ -modules*

$$(4.19) \quad 0 \rightarrow q^{(\alpha_i | 2\Lambda - \beta)} \overline{F}_i M \rightarrow F_i M \rightarrow F_i^\Lambda M \rightarrow 0,$$

which is functorial in  $M \in \text{Mod}(R^\Lambda(\beta))$ .

Now we prove the main theorem of this section.

**Theorem 4.13.** *Set*

$$\begin{aligned} \text{Proj}(R^\Lambda) &= \bigoplus_{\alpha \in \mathbb{Q}^+} \text{Proj}(R^\Lambda(\alpha)), & \text{Rep}(R^\Lambda) &= \bigoplus_{\alpha \in \mathbb{Q}^+} \text{Rep}(R^\Lambda(\alpha)), \quad \text{and} \\ [\text{Proj}(R^\Lambda)] &= \bigoplus_{\alpha \in \mathbb{Q}^+} [\text{Proj}(R^\Lambda(\alpha))], & [\text{Rep}(R^\Lambda)] &= \bigoplus_{\alpha \in \mathbb{Q}^+} [\text{Rep}(R^\Lambda(\alpha))]. \end{aligned}$$

Then the functors  $E_i^\Lambda$  and  $F_i^\Lambda$  are well-defined exact functors on  $\text{Proj}(R^\Lambda)$  and  $\text{Rep}(R^\Lambda)$ , and they induce endomorphisms of the Grothendieck groups  $[\text{Proj}(R^\Lambda)]$  and  $[\text{Rep}(R^\Lambda)]$ .

*Proof.* By Proposition 2.3, Lemma 4.3 and Theorem 4.11,  $F^\Lambda$  is a finitely generated projective module over  $R^\Lambda(\beta)$ . Thus  $F_i^\Lambda$  sends the finite-dimensional left  $R^\Lambda(\beta)$ -modules to finite-dimensional left  $R^\Lambda(\beta + \alpha_i)$ -modules. Similarly,  $e(\beta, i)R^\Lambda(\beta + \alpha_i)$  is a finitely generated projective left  $R^\Lambda(\beta)$ -module and hence  $E_i^\Lambda$  sends finitely generated projective left  $R^\Lambda(\beta + \alpha_i)$ -modules to finitely generated projective left  $R^\Lambda(\beta)$ -modules.  $\square$

The following lemma will be needed in the sequel.

**Lemma 4.14.** *Set*

- $\mathbf{A} = \sum_{\nu \in I^\beta} \mathbf{A}_\nu e(i, \nu),$
- $\mathbf{B} = \sum_{\nu \in I^\beta} a_i^\Lambda(x_{n+1}) \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(x_{n+1}, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(x_{n+1}, x_a) \mathcal{P}_i(x_a, x_{n+1}) e(\nu, i).$

Then we have a commutative diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{P} & K_0 \\ \mathbf{A} \downarrow & \swarrow Q & \downarrow \mathbf{B} \\ K_1 & \xrightarrow{P} & K_0 \end{array}$$

Here the vertical arrows are the multiplication of  $\mathbf{A}$  and  $\mathbf{B}$  from the right, respectively.

*Proof.* We can apply a similar argument given in [7, Lemma 4.19].  $\square$

5. CATEGORIFICATION OF  $V(\Lambda)$ 

In this chapter, we will show that the cyclotomic Khovanov-Lauda-Rouquier algebra  $R^\Lambda$  categorifies the irreducible highest weight  $U_q(\mathfrak{g})$ -module  $V(\Lambda)$ .

**Theorem 5.1.** *For  $i \neq j \in I$ , there exists a natural isomorphism*

$$(5.1) \quad E_i^\Lambda F_j^\Lambda \simeq q_i^{-a_{ij}} F_j^\Lambda E_i^\Lambda.$$

*Proof.* By Corollary 3.4, we already know

$$(5.2) \quad e(n, i)R(n+1)e(n, j) \simeq q_i^{-a_{ij}} R(n)e(n-1, j) \otimes_{R(n-1)} e(n-1, j)R(n).$$

Applying the functor  $R^\Lambda(n) \otimes_{R(n)} \bullet \otimes_{R(n)} R^\Lambda(n)e(\beta)$  on (5.2), we obtain

$$\begin{aligned} & \frac{e(n, i)R(n+1)e(\beta, j)}{e(n, i)R(n)a^\Lambda(x_1)R(n+1)e(\beta, j) + e(n, i)R(n+1)a^\Lambda(x_1)R(n)e(\beta, j)} \\ & \simeq R^\Lambda(n)e(n-1, j) \otimes_{R^\Lambda(n-1)} e(n-1, i)R^\Lambda(n)e(\beta) = F_j^\Lambda E_i^\Lambda R^\Lambda(\beta). \end{aligned}$$

Note that

$$E_i^\Lambda F_j^\Lambda R^\Lambda(\beta) = \left( \frac{e(n, i)R(n+1)e(n, j)}{e(n, i)R(n+1)a^\Lambda(x_1)R(n+1)e(n, j)} \right) e(\beta).$$

Thus it suffices to show that

$$(5.3) \quad \begin{aligned} & e(n, i)R(n+1)a^\Lambda(x_1)R(n+1)e(n, j) \\ & = e(n, i)R(n)a^\Lambda(x_1)R(n+1)e(n, j) + e(n, i)R(n+1)a^\Lambda(x_1)R(n)e(n, j). \end{aligned}$$

Since  $a^\Lambda(x_1)\tau_k = \tau_k a^\Lambda(x_1)$  for all  $k \geq 2$ , we have

$$\begin{aligned} R(n+1)a^\Lambda(x_1)R(n+1) &= \sum_{a=1}^{n+1} R(n+1)a^\Lambda(x_1)\tau_a \cdots \tau_n R(n, 1) \\ &= R(n+1)a^\Lambda(x_1)R(n, 1) + R(n+1)a^\Lambda(x_1)\tau_1 \cdots \tau_n R(n, 1) \\ &= R(n+1)a^\Lambda(x_1)R(n, 1) + \sum_{a=1}^{n+1} R(n, 1)\tau_n \cdots \tau_a a^\Lambda(x_1)\tau_1 \cdots \tau_n R(n, 1) \\ &= R(n+1)a^\Lambda(x_1)R(n, 1) + R(n, 1)a^\Lambda(x_1)R(n+1) + R(n, 1)\tau_n \cdots \tau_1 a^\Lambda(x_1)\tau_1 \cdots \tau_n R(n, 1). \end{aligned}$$

For  $i \neq j$ , we get

$$e(n, i)R(n, 1)\tau_n \cdots \tau_1 a^\Lambda(x_1)\tau_1 \cdots \tau_n R(n, 1)e(n, j) = 0,$$

and our assertion (5.3) follows.  $\square$

**Theorem 5.2.** *Let  $\lambda = \Lambda - \beta$ . Then there exist natural isomorphisms of endofunctors on  $\text{Mod}(R^\Lambda(\beta))$  given below.*

(i) *If  $\langle h_i, \lambda \rangle \geq 0$ , then we have*

$$(5.4) \quad q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i, \lambda \rangle - 1} q_i^{2k} \text{Id} \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda.$$

(ii) *If  $\langle h_i, \lambda \rangle < 0$ , then we have*

$$(5.5) \quad q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda \oplus \bigoplus_{k=0}^{-\langle h_i, \lambda \rangle - 1} q_i^{2k-2} \text{Id}.$$

The rest of this section is devoted to the proof of this theorem.

Consider the following commutative diagram with exact rows and columns derived from Theorem 3.5, Theorem 3.9 and Corollary 4.12:

(5.6)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & q_i^{(\alpha_i | 2\Lambda - \beta)} \overline{F}_i E_i M & \longrightarrow & q_i^{-a_{ii}} F_i E_i M & \longrightarrow & q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & q_i^{(\alpha_i | 2\Lambda - \beta)} E_i \overline{F}_i M & \longrightarrow & E_i F_i M & \longrightarrow & E_i^\Lambda F_i^\Lambda M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & q_i^{(\alpha_i | 2\Lambda - 2\beta)} \mathbf{k}[t_i] \otimes M & \longrightarrow & \mathbf{k}[t_i] \otimes M & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

By taking the kernel modules, we obtain the following commutative diagram of  $(R(\beta), R^\Lambda(\beta))$ -modules:

(5.7)

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & q_i^{(\alpha_i|2\Lambda-\beta)} K'_1 & \xrightarrow{P'} & q_i^{-\alpha_i} K'_0 & \xrightarrow{G} & q_i^{-\alpha_i} F_i^\Lambda E_i^\Lambda R^\Lambda(\beta) \longrightarrow 0 \\
& & \downarrow & & \downarrow F & & \downarrow \\
0 & \longrightarrow & q_i^{(\alpha_i|2\Lambda-\beta)} E_i K_1 & \xrightarrow{P} & E_i K_0 & \longrightarrow & E_i^\Lambda F_i^\Lambda R^\Lambda(\beta) \longrightarrow 0 \\
& & \downarrow B & & \downarrow C & & \\
& & q_i^{(\alpha_i|2\Lambda-2\beta)} \mathbf{k}[t_i] \otimes R^\Lambda(\beta) & \xrightarrow{A} & \mathbf{k}[t_i] \otimes R^\Lambda(\beta) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

where

$$\begin{aligned}
K'_0 &= F_i E_i R^\Lambda(\beta) = R(\beta) e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i) R^\Lambda(\beta) \\
K'_1 &= \overline{F}_i E_i R^\Lambda(\beta) = R(\beta) e(i, \beta - \alpha_i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i) R^1(\beta) \otimes_{R(\beta)} R^\Lambda(\beta) \\
&= R(\beta) e(i, \beta - \alpha_i) \otimes_{R(\beta - \alpha_i)} R^\Lambda(\beta).
\end{aligned}$$

The homomorphisms in the diagram (5.7) can be described as follows:

- $P$  is the right multiplication by  $a^\Lambda(x_1)\tau_1 \cdots \tau_n$  and  $(R(\beta), R^\Lambda(\beta) \otimes \mathbf{k}[t_i])$ -bilinear.
- Similarly,  $P'$  is given by the right multiplication by  $a^\Lambda(x_1)\tau_1 \cdots \tau_{n-1}$  on  $R(\beta) e(i, \beta - \alpha_i)$ .
- The map  $A$  is defined by the chasing the diagram. Note that it is  $R^\Lambda(\beta)$ -linear but *not*  $\mathbf{k}[t_i]$ -linear.
- $B$  is given by taking the coefficient of  $\tau_n \cdots \tau_1$  and  $(R(\beta) \otimes \mathbf{k}[x_{n+1}], \mathbf{k}[x_1] \otimes R^1(\beta))$ -bilinear.
- $F$  is the multiplication by  $\tau_n$  (See Proposition 3.3).
- $C$  is the cokernel map of  $F$ . Thus it is  $(R(\beta), R^\Lambda(\beta))$ -bilinear but does *not* commute with  $t_i$ .
- $G$  is the canonical projection induced from  $P'$ . It is  $(R(\beta) \otimes \mathbf{k}[x_{n+1}], R^\Lambda(\beta) \otimes \mathbf{k}[x_{n+1}])$ -bilinear.

Set  $p = \text{Supp}_i(\beta)$ . Note that the degree of  $t_i$  in

$$\prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(t_i, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(t_i, x_{a+1}) \mathcal{P}_i(x_{a+1}, t_i)$$

is given by

$$-\langle h_i, \beta - p\alpha_i \rangle + 2p(1 - \frac{a_{ii}}{2}) = -\langle h_i, \beta \rangle + pa_{ii} + 2p - pa_{ii} = -\langle h_i, \beta \rangle + 2p.$$

Define an invertible element  $\gamma \in \mathbf{k}^\times$  by

$$\begin{aligned} (-1)^p \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(t_i, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(t_i, x_{a+1}) \mathcal{P}_i(x_{a+1}, t_i) \\ (5.8) \quad = \gamma^{-1} t_i^{-\langle h_i, \beta \rangle + 2p} + (\text{terms of degree } < -\langle h_i, \beta \rangle + 2p \text{ in } t_i). \end{aligned}$$

Set  $\lambda = \Lambda - \beta$  and

$$(5.9) \quad \varphi_k = A(t_i^k) \in \mathbf{k}[t_i] \otimes R^\Lambda(\beta),$$

which is of degree  $2(\alpha_i | \lambda) + 2d_i k = 2d_i(\langle h_i, \lambda \rangle + k)$ .

The following proposition is one of the key ingredients of the proof of Theorem 5.2

**Proposition 5.3.** *If  $\langle h_i, \lambda \rangle + k \geq 0$ , then  $\gamma\varphi_k$  is a monic polynomial in  $t_i$  of degree  $\langle h_i, \lambda \rangle + k$ .*

Note that for  $m < 0$ , we say that a polynomial  $\varphi$  is a monic polynomial of degree  $m$  if  $\varphi = 0$ .

To prove Proposition 5.3, we need some preparation. Let

$$z = \sum_{k \in \mathbb{Z}_{>0}} a_k \otimes b_k \in R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i, i)R^\Lambda(\beta),$$

where  $a_k \in R(\beta)e(\beta - \alpha_i, i)$  and  $b_k \in e(\beta - \alpha_i, i)R^\Lambda(\beta)$ . Define a map  $E: K'_0 \rightarrow E_i K_0$  by

$$(5.10) \quad z \mapsto \sum_{k \in \mathbb{Z}_{>0}} a_k \mathcal{P}_i(x_n, x_{n+1}) b_k.$$

**Lemma 5.4.** *For  $z \in R(\beta)e(\beta - \alpha_i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i, i)R^\Lambda(\beta)$ , we have*

$$(5.11) \quad F(z)x_{n+1} = F(z(x_n \otimes 1)) + E(z).$$

*Proof.* Let  $z = a \otimes b \in R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i, i)R^\Lambda(\beta)$ , where  $a \in R(\beta)e(\beta - \alpha_i, i)$  and  $b \in e(\beta - \alpha_i, i)R^\Lambda$ . Then

$$F(z) = a\tau_n b, \quad E(z) = a\mathcal{P}_i(x_n, x_{n+1})b.$$

Thus

$$\begin{aligned} F(z)x_{n+1} &= a\tau_n bx_{n+1} = a\tau_n x_{n+1}b = a(x_n\tau_n + \mathcal{P}_i(x_n, x_{n+1}))b \\ &= ax_n\tau_n b + a\mathcal{P}_i(x_n, x_{n+1})b \\ &= F(ax_n \otimes b) + E(z) = F(z(x_n \otimes 1)) + E(z). \end{aligned}$$

□

By Proposition 3.3, we have

$$(5.12) \quad \begin{aligned} &e(\beta, i)R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} R^\Lambda(\beta) \\ &= F(R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i, i)R^\Lambda(\beta)) \oplus (R^\Lambda(\beta) \otimes \mathbf{k}[t_i])e(\beta, i), \end{aligned}$$

where  $t_i = x_{n+1}$ . Using the decomposition (5.12), we write

$$(5.13) \quad P(e(\beta, i)\tau_n \cdots \tau_1 x_1^k e(i, \beta)) = F(\psi_k) + \varphi_k$$

for uniquely determined  $\psi_k \in K'_0$  and  $\varphi_k \in \mathbf{k}[t_i] \otimes R^\Lambda(\beta)$ .

Using (5.9), we have

$$A(t_i^k) = AB(e(\beta, i)\tau_n \cdots \tau_1 x_1^k e(i, \beta)) = CP(e(\beta, i)\tau_n \cdots \tau_1 x_1^k e(i, \beta)) = \varphi_k.$$

Thus one can verify that the definition of  $\varphi_k$  coincides with the definition given in (5.9).

Since

$$\begin{aligned} F(\psi_{k+1}) + \varphi_{k+1} &= P(e(\beta, i)\tau_n \cdots \tau_1 x_1^{k+1} e(i, \beta)) = P(e(\beta, i)\tau_n \cdots \tau_1 x_1^k e(i, \beta))x_{n+1} \\ &= (F(\psi_k) + \varphi_k)x_{n+1} = F(\psi_k(x_n \otimes 1)) + E(\psi_k) + \varphi_k t_i, \end{aligned}$$

we have

$$(5.14) \quad \psi_{k+1} = \psi_k(x_n \otimes 1), \quad \varphi_{k+1} = E(\psi_k) + \varphi_k t_i.$$

Now we will prove Proposition 5.3. By Lemma 4.14, we have

$$g_n \cdots g_1 x_1^k e(i, \nu) \tau_1 \cdots \tau_n = x_{n+1}^k a_i^\Lambda(x_{n+1}) \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(x_{n+1}, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(x_{n+1}, x_a) \mathcal{P}_i(x_a, x_{n+1}) e(\nu, i)$$

in  $e(\beta, i)R(\beta + \alpha_i)e(\beta, i) \otimes R^\Lambda(\beta)$ , which implies

$$\begin{aligned} AB(g_n \cdots g_1 x_1^k e(i, \nu)) &= C \left( x_{n+1}^k a_i^\Lambda(x_{n+1}) \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(x_{n+1}, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(x_{n+1}, x_a) \mathcal{P}_i(x_a, x_{n+1}) \right) e(\nu, i) \\ &= t_i^k a_i^\Lambda(t_i) \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(t_i, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(t_i, x_a) \mathcal{P}_i(x_a, t_i) e(\nu). \end{aligned}$$

On the other hand, since  $B$  is the map taking the coefficient of  $\tau_n \cdots \tau_1$ , we have

$$\begin{aligned} B(g_n \cdots g_1 x_1^k e(i, \nu)) &= B \left( \prod_{\nu_a = i} (-(x_{n+1} - x_a)^2) x_{n+1}^k e(\nu, i) \tau_n \cdots \tau_1 \right) \\ &= t_i^k \prod_{\nu_a = i} (-(t_i - x_a)^2) e(\nu). \end{aligned}$$

Thus we have

$$(5.15) \quad A(t_i^k \prod_{\nu_a = i} (t_i - x_a)^2 e(\nu)) = (-1)^p t_i^k a_i^\Lambda(t_i) \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(t_i, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(t_i, x_a) \mathcal{P}_i(x_a, t_i) e(\nu).$$

Set

$$\begin{aligned} S &= \sum_{\nu \in I^\beta} \prod_{\nu_a = i} (t_i - x_a)^2 e(\nu) \in \mathbf{k}[t_i] \otimes R^\Lambda(\beta), \\ F &= \gamma(-1)^p a_i^\Lambda(t_i) \sum_{\nu \in I^\beta} \left( \prod_{\substack{1 \leq a \leq n, \\ \nu_a \neq i}} \mathcal{Q}_{i, \nu_a}(t_i, x_a) \prod_{\substack{1 \leq a \leq n, \\ \nu_a = i}} \mathcal{P}_i(t_i, x_a) \mathcal{P}_i(x_a, t_i) e(\nu) \right) \in \mathbf{k}[t_i] \otimes R^\Lambda(\beta). \end{aligned}$$

Then they are monic polynomials in  $t_i$  of degree  $2p$  and  $\langle h_i, \lambda \rangle + 2p$ , respectively ( $p := \text{Supp}_i(\beta)$ ). Note that they are contained in the center of  $\mathbf{k}[t_i] \otimes R^\Lambda(\beta)$ . Then (5.15) can be expressed as the following form:

$$(5.16) \quad \gamma A(t_i^k S) = t_i^k F.$$

Note that

- if  $\text{Supp}_i(\beta) = 0$ , then  $K'_0 = 0$  and
- if  $i \in I^{\text{im}}$  such that  $\langle h_i, \lambda \rangle = 0$  and  $\text{Supp}_i(\beta) > 0$ , then  $R^\Lambda(\beta) = 0$  (see Lemma 4.4).

Thus, to prove Proposition 5.3, we may assume that

$$(5.17) \quad \begin{aligned} &\text{Supp}_i(\beta) > 0 \text{ and} \\ &\text{if } i \in I^{\text{im}}, \text{ then } \langle h_i, \lambda \rangle > 0. \end{aligned}$$

**Lemma 5.5.** *For any  $k \geq 0$ , we have*

$$(5.18) \quad t_i^k \mathbf{F} = (\gamma \varphi_k) \mathbf{S} + \mathbf{h}_k,$$

where  $\mathbf{h}_k \in R^\Lambda(\beta)[t_i]$  is a polynomial in  $t_i$  of degree  $< 2p$ . In particular,  $\gamma \varphi_k$  coincides with the quotient of  $t_i^k \mathbf{F}$  by  $\mathbf{S}$ .

*Proof.* By (5.14),  $A(t_i^{k+1}) - A(t_i^k)t_i \in R^\Lambda(\beta)$ , which implies

$$(5.19) \quad A(at_i) - A(a)t_i \in R^\Lambda(\beta)[t_i] \text{ is of degree } \leq 0 \text{ in } t_i \text{ for any } a \in R^\Lambda(\beta)[t_i].$$

We will show

$$(5.20) \quad \begin{aligned} &\text{for any polynomial } f \in R^\Lambda(\beta)[t_i] \text{ in } t_i \text{ of degree } m \text{ and } a \in R^\Lambda(\beta)[t_i], \\ &A(af) - A(a)f \text{ is of degree } < m. \end{aligned}$$

We will use induction on  $m$ . By the fact that  $A$  is  $R^\Lambda(\beta)$ -linear and (5.19), it holds for  $m = 0$  and 1. Thus it suffices to show (5.20) when  $f = t_i g$  and (5.20) is true for  $g$ . Then

$$A(af) - A(a)f = (A(at_i g) - A(at_i)g) + (A(at_i) - A(a)t_i)g.$$

Then the first term is of degree  $< \deg(g)$  in  $t_i$  and the second term is of degree  $< \deg(g) + 1$ . Hence we prove (5.20). Thus we have

$$t_i^k \gamma^{-1} \mathbf{F} - \varphi_k \mathbf{S} = t_i^k \gamma^{-1} \mathbf{F} - A(t_i^k) \mathbf{S} = A(t_i^k \mathbf{S}) - A(t_i^k) \mathbf{S}$$

by (5.16) and it is of degree  $< 2p$  by applying (5.20) for  $f = \mathbf{S}$ .  $\square$

Thus by Lemma 5.5, we can conclude that  $\gamma \varphi_k$  is a monic polynomial in  $t_i$  of degree  $\langle h_i, \lambda \rangle + k$ , which completes the proof of Proposition 5.3.

**Proof of Theorem 5.2:** By the Snake Lemma, we have the following exact sequence

$$0 \rightarrow \text{Ker} A \rightarrow q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda R^\Lambda(\beta) \rightarrow E_i^\Lambda F_i^\Lambda R^\Lambda(\beta) \rightarrow \text{Coker} A \rightarrow 0.$$

If  $\langle h_i, \lambda \rangle \geq 0$ , by Proposition 5.3, we have

$$\text{Ker} A = 0, \quad \bigoplus_{k=0}^{\langle h_i, \lambda \rangle - 1} \mathbf{k} t_i^k \otimes R^\Lambda(\beta) \xrightarrow{\sim} \text{Coker} A.$$



Hence we obtain

$$q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i, \lambda \rangle - 1} q_i^{2k} \text{Id} \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda,$$

which is the proof the statement of Theorem 5.2 (1).

If  $\langle h_i, \lambda \rangle < 0$ , then  $i \in I^{\text{re}}$ . In this case, the proof is the same as in [7, Theorem 5.2 (b)].  $\square$

We define the modified functors  $\mathcal{E}_i^\Lambda$  and  $\mathcal{F}_i^\Lambda$  on  $\text{Mod}(R)$ :

$$\mathcal{E}_i^\Lambda = E_i^\Lambda, \quad \mathcal{F}_i^\Lambda = q_i^{1 - \langle h_i, \Lambda - \beta \rangle} F_i^\Lambda.$$

Then by applying degree shift functor  $q_i^{1 - \langle h_i, \Lambda - \beta \rangle}$  to the equations (5.1), (5.4) and (5.5), we obtain the natural isomorphisms

$$(5.21) \quad \begin{aligned} \mathcal{E}_i^\Lambda \mathcal{F}_j^\Lambda &\simeq \mathcal{F}_j^\Lambda \mathcal{E}_i^\Lambda && \text{if } i \neq j, \\ \mathcal{E}_i^\Lambda \mathcal{F}_i^\Lambda &\simeq \mathcal{F}_i^\Lambda \mathcal{E}_i^\Lambda \oplus \frac{q_i^{\langle h_i, \Lambda - \beta \rangle} - q_i^{-\langle h_i, \Lambda - \beta \rangle}}{q_i - q_i^{-1}} \text{Id} && \text{if } \langle h_i, \Lambda - \beta \rangle \geq 0, \\ \mathcal{E}_i^\Lambda \mathcal{F}_i^\Lambda &\oplus \frac{q_i^{-\langle h_i, \Lambda - \beta \rangle} - q_i^{\langle h_i, \Lambda - \beta \rangle}}{q_i - q_i^{-1}} \text{Id} \simeq \mathcal{F}_i^\Lambda \mathcal{E}_i^\Lambda && \text{if } \langle h_i, \Lambda - \beta \rangle \leq 0 \end{aligned}$$

on  $\text{Mod}(R^\Lambda(\beta))$ . Now, assume that  $\mathbf{k}_0$  is a field. Then, as operators on  $[\text{Proj}(R^\Lambda)]$  and  $[\text{Rep}(R^\Lambda)]$ , they satisfy the commutation relations

$$[\mathcal{E}_i^\Lambda, \mathcal{F}_j^\Lambda] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

where

$$K_i|_{[\text{Proj}(R^\Lambda(\beta))]} := q_i^{\langle h_i, \Lambda - \beta \rangle}, \quad K_i|_{[\text{Rep}(R^\Lambda(\beta))]} := q_i^{\langle h_i, \Lambda - \beta \rangle}.$$

Combining Lemma 4.3, Lemma 4.4 and Theorem 2.4 as in [7, Section 6], we obtain a categorification of the irreducible highest weight  $U_q(\mathfrak{g})$ -module  $V(\Lambda)$ :

**Theorem 5.6.** *If  $a_{ii} \neq 0$  for all  $i \in I$ , then there exist  $U_{\mathbb{A}}(\mathfrak{g})$ -module isomorphisms*

$$[\text{Proj}(R^\Lambda)] \simeq V_{\mathbb{A}}(\Lambda) \quad \text{and} \quad [\text{Rep}(R^\Lambda)] \simeq V_{\mathbb{A}}(\Lambda)^\vee.$$

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